

LACK OF DIAMAGNETISM AND THE LITTLE-PARKS EFFECT

SØREN FOURNAIS AND MIKAEL PERSSON SUNDQVIST

ABSTRACT. When a superconducting sample is submitted to a sufficiently strong external magnetic field, the superconductivity of the material is lost. In this paper we prove that this effect does not, in general, take place at a unique value of the external magnetic field strength. Indeed, for a sample in the shape of a narrow annulus the set of magnetic field strengths for which the sample is superconducting is not an interval. This is a rigorous justification of the Little–Parks effect. We also show that the same oscillation effect can happen for disc-shaped samples if the external magnetic field is non-uniform. In this case the oscillations can even occur repeatedly along arbitrarily large values of the Ginzburg–Landau parameter κ . The analysis is based on an understanding of the underlying spectral theory for a magnetic Schrödinger operator. It is shown that the ground state energy of such an operator is not in general a monotone function of the intensity of the field, even in the limit of strong fields.

1. INTRODUCTION

1.1. Discussion. We will consider the Ginzburg–Landau model of superconductivity. If a 2-dimensional superconducting sample with Ginzburg–Landau parameter κ is submitted to a uniform magnetic field of strength σ , then (by a theorem of Giorgi and Phillips [12]) there exists a field strength $\overline{H_{C_3}(\kappa)}$ such that if $\sigma > \overline{H_{C_3}(\kappa)}$, then the sample will be in its normal state, i.e. superconductivity is lost altogether. It is at first sight natural to expect this phenomenon to mark a monotone transition, i.e. to expect that the material is in its superconducting (possibly mixed) state for all $\sigma < \overline{H_{C_3}(\kappa)}$.

Indeed, such a monotonicity result has been proved recently in a number of geometric situations and in both 2 and 3 dimensional settings [5, 6, 7, 9] in the case where the Ginzburg–Landau parameter κ is large (it also follows from asymptotic expansions obtained in other works such as [18, 3]). However, Nature does not support this monotonicity in general. The famous Little–Parks effect [16] shows that for narrow cylinders (or 2D annuli) one has an oscillatory behavior instead of monotonicity.¹

In this paper we will establish such ‘oscillatory’ effects rigorously in different geometric settings.

The lack of monotonicity comes from the topology/geometry of the annulus. It is natural to ask whether one can get such an oscillatory effect for (non-vanishing) magnetic fields defined on domains without topology. From the previous investigations [5] we know this to be impossible for a uniform magnetic field, but how

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¹In connection to the Little–Parks effect one often discusses the (solid) disc as another example, where the effect of surface superconductivity provides a localization to the boundary and therefore effectively introduces non-trivial topology which should give oscillations. However, as already the early studies of Saint-James [20] show (see also [6]), in the case of the solid disc these oscillations are superposed on a linear background and are not strong enough to break the monotonicity of the background.

about more general fields? The analysis of constant magnetic fields tells us that this question is linked to a purely spectral problem, namely whether the first eigenvalue of the Schrödinger operator $(-i\nabla + B\mathbf{F})^2$ is monotone increasing in the parameter (strength of the magnetic field) B for sufficiently large values of B . This property has been called ‘strong diamagnetism’ and has been proved for large classes of magnetic fields—it is even ‘generically’ satisfied [5, 6, 7, 9, 18, 3]. However, we produce counterexamples in the general case.

1.2. Ginzburg–Landau theory. The Ginzburg–Landau theory of superconductivity is based on the energy functional

$$\begin{aligned} \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) = & \int_{\Omega} |(-i\nabla + \kappa\sigma\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 dx \\ & + (\kappa\sigma)^2 \int_{\tilde{\Omega}} |\operatorname{curl} \mathbf{A} - \beta|^2 dx. \end{aligned}$$

Here $\kappa > 0$ is a material parameter (the Ginzburg–Landau parameter), $\sigma \geq 0$ is a parameter measuring the intensity of the external magnetic field. The domain $\Omega \subseteq \mathbb{R}^2$ is the part of space occupied by the superconducting material. For $\tilde{\Omega}$ there are two natural choices. One can take $\tilde{\Omega} = \mathbb{R}^2$. That will not be our choice here because for reasons of simplicity we want to avoid an unnecessary technical complication connected with unbounded domains in \mathbb{R}^2 (for details on how to handle this issue see [11, 13]). One can also—and that will be our convention here—take $\tilde{\Omega}$ to be the smallest simply connected domain containing Ω , i.e. the union of Ω and all the ‘holes’ in Ω . The function $\beta \in L^2(\tilde{\Omega})$ is the profile of the external magnetic field.

In the setting of bounded $\Omega \subset \mathbb{R}^2$ the functional $\mathcal{G}_{\kappa,\sigma}$ is naturally defined on $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1(\tilde{\Omega}, \mathbb{R}^2)$. The functional is immediately seen to be gauge invariant, $\mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) = \mathcal{G}_{\kappa,\sigma}(\psi e^{-i\kappa\sigma\varphi}, \mathbf{A} + \nabla\varphi)$. The vector field \mathbf{A} models the induced magnetic vector potential. The function ψ measures the superconducting properties of the material, with $|\psi(x)|$ being a measure of the local density of Cooper pairs.

We say that a minimizer (ψ, \mathbf{A}) of the Ginzburg–Landau functional is trivial if $\psi \equiv 0$ and $\operatorname{curl} \mathbf{A} = \beta$. In each of the situations we will encounter, the notation \mathbf{F} will be reserved for a fixed choice of vector potential with $\operatorname{curl} \mathbf{F} = \beta$. For trivial minimizers we clearly have $\mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) = 0$. For a nontrivial minimizer the functional must be negative, since one gets from the Euler-Lagrange equations of a minimizer that

$$\mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) = -\frac{\kappa^2}{2} \|\psi\|_4^4,$$

if (ψ, \mathbf{A}) is a minimizer.

We define the set

$$\mathcal{N}(\kappa) := \{\sigma > 0 \mid \mathcal{G}_{\kappa,\sigma} \text{ has a nontrivial minimizer } (\psi, \mathbf{A})\}.$$

Following [17] one typically defines the third critical field to be given by $\sup \mathcal{N}(\kappa)$, which is finite by [12]. However, unless $\mathcal{N}(\kappa)$ is an interval, this definition is not the only natural one to take—see [5, 8] for a discussion. We will see below that this is not always the case.

1.3. Oscillations in the third critical field. Let $\Omega = \{x \in \mathbb{R}^2 \mid R_i < |x| < R_o\}$ denote the annulus with inner radius R_i and outer radius R_o , let $\beta \equiv 1$. In this case we will write $D = \tilde{\Omega} = B(0, R_o)$ i.e. the disc of radius R_o .

Theorem 1.1. *There exists an annulus Ω and a $\kappa_0 > 0$ such the set $\mathcal{N}(\kappa_0)$ is not an interval.*

Remark 1.2. The mechanism behind this result is a convergence of the magnetic quadratic form on the annulus to the corresponding form on the circle. This convergence was already noticed in the works [2, 19], where also ‘annuli’ of non-uniform width were considered. It is likely that one could deduce Theorem 1.1 from these works, however, we prefer to give a simple independent proof which also emphasizes the connection to the Bohm–Aharonov-effect.

Remark 1.3. By shrinking the inner radius R_i of the annulus, we can get κ_0 as large as we want, since the eigenvalues of the limiting problem will then cross at a level $1/(2R_i)^2$. In particular it is possible to have $\kappa_0 > 1/\sqrt{2}$, which means that Theorem 1.1 also applies to superconductors of Type II.

One may criticize the result of Theorem 1.1 on two accounts. One could desire not to have the topology fixed a priori, but rather have it generated by localization properties of the minimizer. Also most previous mathematical analysis has considered the limit of large values of κ . One can show that for sufficiently large values of κ the set $\mathcal{N}(\kappa)$ of a superconducting sample in the shape of an annulus will behave as the one of the disc with the same outer radius, and it is known that for the disc and with constant magnetic field—for sufficiently large values of κ — $\mathcal{N}(\kappa)$ is indeed an interval [5].

Our next theorem remedies these defects.

Theorem 1.4. *Let Ω be the unit disc in \mathbb{R}^2 . There exists an everywhere positive magnetic field $\beta(x)$ such that for all $\kappa_0 > 0$ there exists $\kappa > \kappa_0$ satisfying that $\mathcal{N}(\kappa)$ is not an interval.*

In fact, the magnetic field can be chosen as $\beta(x) = \delta + (1 - |x|)^2$, where $\delta > 0$ is some sufficiently small constant. Theorem 1.4 follows directly from Theorem 1.8 (or Theorem 1.12) below using [8, Prop. 13.1.7]. Actually, it easily follows from Theorem 1.12 below, that for all integers $n > 0$ we can choose δ so small that $\mathcal{N}(\kappa)$ will consist of at least n intervals for all κ sufficiently large.

1.4. Lack of strong diamagnetism.

For easy reference we collect the notation and assumptions concerning the magnetic fields that we will treat. We will work on an open set Ω being one of the following three cases $\Omega \in \{\mathbb{R}^2, B(0, 1), \mathbb{R}^2 \setminus \overline{B(0, 1)}\}$.

Assumption 1.5. Suppose that $\beta(x) = \tilde{\beta}(|x|) \in L_{\text{loc}}^\infty(\Omega)$, is a non-negative, radial magnetic field, possessing five continuous derivatives in an open neighborhood U of the unit circle $\{x \in \mathbb{R}^2 : |x| = 1\}$. Define

$$\delta := \tilde{\beta}(1) \geq 0,$$

and assume that $\tilde{\beta}'(1) = 0$ and write

$$\tilde{\beta}''(1) =: k.$$

When $\Omega \in \{B(0, 1), \mathbb{R}^2 \setminus \overline{B(0, 1)}\}$, we assume that

$$\Theta_0 \delta < \inf_{x \in \Omega} \beta(x),$$

where $\Theta_0 < 1$ is the spectral constant recalled in Appendix A. When $\Omega = \mathbb{R}^2$, we impose the stronger assumption that $\tilde{\beta}(r)$ has a unique, non-degenerate minimum at $r = 1$ and that

$$\inf_{x \in \mathbb{R}^2 \setminus U} \beta(x) > \delta.$$

Remark 1.6. The assumptions assure that ground state eigenfunctions will be localized near $r = 1$. For $\Omega = \mathbb{R}^2$, we have $k > 0$ by assumption, but that is not necessarily true in the cases with boundary.

Definition 1.7. We define

$$\Phi := \frac{1}{2\pi} \int_{\{|x|<1\}} \beta(x) dx = \int_0^1 \tilde{\beta}(r) r dr,$$

i.e. Φ denotes the magnetic flux through the unit disc.

For a magnetic field satisfying Assumption 1.5 and $B > 0$, we study the lowest eigenvalue $\lambda_{1,\mathcal{H}(B)}$ of the self-adjoint magnetic Schrödinger operator

$$\mathcal{H}(B) = (-i\nabla + B\mathbf{F})^2$$

in $L^2(\Omega)$. Here \mathbf{F} is a magnetic vector potential associated with the magnetic field β . We refer the reader to Section 2 for a more complete definition of this operator and the eigenvalue.

We will study this eigenvalue problem in three cases, namely for Ω the unit disc, the complement of the unit disc and the whole plane \mathbb{R}^2 . If Ω has a non-empty boundary we impose a magnetic Neumann boundary condition.

The next theorem states that if Ω is the unit disc or its complement, then special choices of magnetic fields satisfying Assumption 1.5 will give that the function $B \mapsto \lambda_{1,\mathcal{H}(B)}$ is *not* monotonically increasing for large B . Before stating the theorems, we remind the reader that

$$\xi_0, \quad \Theta_0, \quad \text{and} \quad \varphi_{\xi_0}(0)$$

are universal (spectral) constants coming from the de Gennes model operator—this is recalled in Appendix A.

Theorem 1.8. *Let Ω be the unit disc or its complement. Suppose that β satisfies Assumption 1.5. Assume that $\delta > 0$ and*

$$\Phi > \frac{\Theta_0}{\xi_0 \varphi_{\xi_0}(0)^2} \delta. \tag{1.1}$$

Then for all $B_0 > 0$ there exist B_1 and B_2 , with $B_0 < B_1 < B_2$, such that

$$\lambda_{1,\mathcal{H}(B_1)} > \lambda_{1,\mathcal{H}(B_2)}.$$

On the other hand, if

$$\Phi < \frac{\Theta_0}{\xi_0 \varphi_{\xi_0}(0)^2} \delta.$$

Then there exists $B_0 > 0$ such that $B \mapsto \lambda_{1,\mathcal{H}(B)}$ is monotone increasing on $[B_0, \infty)$.

Remark 1.9. In particular, (1.1) holds for the magnetic field

$$\beta(x) = \delta + (1 - |x|)^2,$$

for all $\delta > 0$ sufficiently small—the flux in this case is $\Phi = \frac{\delta}{2} + \frac{1}{12}$. Therefore, this magnetic field will not display monotonicity for large field strength.

Theorem 1.8 is a consequence of the following precise asymptotic formulas for the ground state eigenvalue given as Theorem 1.10 and Theorem 1.12.

Theorem 1.10. *Suppose that Ω is the complement of the unit disc, that β satisfies Assumption 1.5 and that $\delta > 0$. Then there are constants C_0^{ext} and C_1^{ext} such that if*

$$\Delta_B^{ext} := \inf_{m \in \mathbb{Z}} |m - \Phi B - \xi_0(\delta B)^{1/2} - C_0^{ext}|,$$

then, as $B \rightarrow +\infty$,

$$\lambda_{1,\mathcal{H}(B)} = \Theta_0 \delta B + \frac{1}{3} \varphi_{\xi_0}(0)^2 (\delta B)^{1/2} + \xi_0 \varphi_{\xi_0}(0)^2 ((\Delta_B^{ext})^2 + C_1^{ext}) + \mathcal{O}(B^{-1/2}).$$

Remark 1.11. By a careful reading of the proof, one will realize that the constant C_0^{ext} is independent of δ but that C_1^{ext} depends on δ . However, for our purposes this extra information is irrelevant.

A similar expansion holds in the interior of the unit disc.

Theorem 1.12. *Suppose that Ω is the unit disc, that β satisfies Assumption 1.5 and that $\delta > 0$. Then there exist constants C_0^{int} and C_1^{int} such that if*

$$\Delta_B^{\text{int}} := \inf_{m \in \mathbb{Z}} |m - \Phi B + \xi_0(\delta B)^{1/2} - C_0^{\text{int}}|,$$

then, as $B \rightarrow +\infty$,

$$\lambda_{1, \mathcal{H}(B)} = \Theta_0 \delta B - \frac{1}{3} \varphi_{\xi_0}(0)^2 (\delta B)^{1/2} + \xi_0 \varphi_{\xi_0}(0)^2 ((\Delta_B^{\text{int}})^2 + C_1^{\text{int}}) + \mathcal{O}(B^{-1/2}).$$

Remark 1.13. Notice that for the disc or its complement, the constant magnetic field $\beta(x) = \delta > 0$ satisfies Assumption 1.5, so Theorems 1.10 and 1.12 imply this special case. This agrees with the calculations in [5] (see also [8]). In the case of constant field (1.1) is not satisfied, and one does get monotonicity of the ground state energy for large magnetic field (this is discussed in detail in [5]).

We continue with $\Omega = \mathbb{R}^2$. Here, we are only able to destroy monotonicity in the case $\delta = 0$.

Theorem 1.14. *Let $\Omega = \mathbb{R}^2$. Then, for all $\delta > 0$ and all magnetic fields satisfying Assumption 1.5 there exists a $B_0 > 0$ such that $\lambda_{1, \mathcal{H}(B)}$ is monotonically increasing for $B > B_0$. However, if $\delta = 0$, then $B \mapsto \lambda_{1, \mathcal{H}(B)}$ is not monotone increasing on any unbounded half-interval.*

As for the disc and the exterior of the disc, the proof of this result goes via asymptotic expansions.

Theorem 1.15. *Suppose that $\Omega = \mathbb{R}^2$, and that β satisfies Assumption 1.5 with $\delta > 0$. Then, as $B \rightarrow +\infty$,*

$$\lambda_{1, \mathcal{H}(B)} = \delta B + \frac{k}{4\delta} + \mathcal{O}(B^{-1/2}).$$

Theorem 1.16. *Let $c_0 > 0$ and Ξ be the spectral constants from (B.1) and (B.2) respectively. Suppose that $\Omega = \mathbb{R}^2$, and that β satisfies Assumption 1.5 with $\delta = 0$. There exist constants C_1 and C_2 such that if*

$$\Delta_B := \inf_{m \in \mathbb{Z}} |m - \Phi B - C_1|,$$

then, as $B \rightarrow +\infty$,

$$\lambda_{1, \mathcal{H}(B)} = \left(\frac{k}{2}\right)^{1/2} \Xi B^{1/2} + \frac{c_0}{2} (\Delta_B^2 + C_2) + o(1).$$

Remark 1.17. In all of the results above the ground state has angular momentum $m \approx \Phi B$ (to leading order in B). We recall that ΦB is the total flux through the unit disc—the bounded domain enclosed by the curve where we have localization. The possibility to obtain non-monotonicity comes from the condition that m must be an integer, which leads to frustration. This is similar to examples in [4].

Remark 1.18. Theorem 1.14 raises the question whether one can break strong diamagnetism with a strictly positive magnetic field on the whole plane.

1.5. Organization of the paper. In the next section we define the operators involved and perform the Fourier decomposition reducing the study to a family of ordinary differential operators.

In Section 3 we prove a non-monotonicity result for an annulus and use that to prove Theorem 1.1. In Section 4 we work in the exterior of the unit disc and prove Theorem 1.10. We indicate in Section 5 how the proof of Theorem 1.10 can be modified to give the proof of Theorem 1.12. In Section 6 we see how Theorem 1.10 and Theorem 1.12 imply Theorem 1.8.

In Section 7 we prove Theorem 1.15 and in Section 8 we prove Theorem 1.16. These two results are used to prove Theorem 1.14.

2. PRELIMINARIES

2.1. Definition of the operator. We consider the self-adjoint magnetic Neumann Schrödinger operator

$$\mathcal{H}(B) = (-i\nabla + B\mathbf{F})^2$$

with domain

$$\begin{aligned} \text{Dom}(\mathcal{H}(B)) = \{ \psi \in L^2(\Omega) \mid & (-i\nabla + B\mathbf{F})^2 \psi \in L^2(\Omega) \\ & \text{and } N(x) \cdot (-i\nabla + B\mathbf{F})\psi|_{\partial\Omega} = 0 \}. \end{aligned}$$

Here $N(x)$ is the interior unit normal to $\partial\Omega$,

$$\beta(x) = \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right), \quad \mathbf{F} = (F_1, F_2),$$

and $B \geq 0$ is the strength of the magnetic field.

In general, for a self-adjoint operator \mathcal{H} that is semi-bounded from below we will write

$$\lambda_{1,\mathcal{H}} = \inf \text{Spec}(\mathcal{H})$$

for the lowest point of the spectrum of \mathcal{H} .

In the case of the disc or if $\beta(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$ the operator has compact resolvent (see [1]). If Ω is unbounded and if $\beta(x) \not\rightarrow +\infty$, then the essential spectrum will be bounded below by $\liminf_{r \rightarrow +\infty} B\tilde{\beta}(r) > B\delta$ (see [14] for the case of \mathbb{R}^2 and [15] for the case of the exterior of the disc). In any case, as it will follow by the results below, $\lambda_{1,\mathcal{H}(B)}$ will be an eigenvalue.

2.2. Fourier decomposition. We will work in domains Ω that are rotationally symmetric. For that reason, we will often work in polar coordinates

$$\begin{cases} x_1 = r \cos \theta, \\ x_2 = r \sin \theta, \end{cases} \quad r \in I, \quad 0 \leq \theta < 2\pi.$$

Here $I \subset [0, +\infty)$ will be an interval.

Moreover, we will work with magnetic fields that depends only on $r = |x|$.

For a radial magnetic field $\beta(x) = \tilde{\beta}(r)$ we will work with the gauge

$$\mathbf{F}(x) = a(r)(-\sin \theta, \cos \theta),$$

where²

$$a(r) = \frac{1}{r} \int_0^r \tilde{\beta}(s) s \, ds. \tag{2.1}$$

²Notice that $\int_0^r \tilde{\beta}(s) s \, ds = \frac{1}{2\pi} \int_{B(0,r)} \beta(x) \, dx$, so $ra(r)$ has an immediate interpretation in terms of the flux through the disc of radius r .

In calculations, we will often meet the expression $(\frac{m}{r} - Ba(r))^2$. This we can write as

$$\left(\frac{m}{r} - Ba(r)\right)^2 = \frac{1}{r^2} (m - Bra(r))^2, \quad (2.2)$$

where

$$ra(r) = \int_0^1 \tilde{\beta}(s)s \, ds + \int_1^r \tilde{\beta}(s)s \, ds = \Phi + \int_0^{r-1} \tilde{\beta}(1+s)(1+s) \, ds. \quad (2.3)$$

Thus, under Assumption 1.5, as $r \rightarrow 1$,

$$ra(r) = \Phi + \delta(r-1) + \frac{\delta}{2}(r-1)^2 + \frac{k}{6}(r-1)^3 + \left(\frac{c}{24} + \frac{k}{8}\right)(r-1)^4 + \mathcal{O}((r-1)^5). \quad (2.4)$$

with $c = \tilde{\beta}'''(1)$.

The expression for the operator $\mathcal{H}(B)$ in polar coordinates becomes

$$\mathcal{H}(B) = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \left(\frac{i}{r} \frac{\partial}{\partial \theta} - Ba(r)\right)^2.$$

We decompose the Hilbert space as (Here I denotes any of the intervals (R_i, R_o) , $(0, 1)$, $(1, +\infty)$ or $(0, +\infty)$)

$$L^2(\Omega) \cong L^2(I, r dr) \otimes L^2(\mathbb{S}^1, d\theta) \cong \bigoplus_{m=-\infty}^{\infty} L^2(I, r dr) \otimes \frac{e^{-im\theta}}{\sqrt{2\pi}},$$

that is, for a function $\psi \in L^2(\Omega)$, we write

$$\psi(r, \theta) = \sum_{m \in \mathbb{Z}} \psi_m(r) \frac{e^{-im\theta}}{\sqrt{2\pi}},$$

where $\psi_m \in L^2(I, r dr)$. Next, we write the operator $\mathcal{H}(B)$ corresponding to this decomposition as

$$\mathcal{H}(B) = \bigoplus_{m=-\infty}^{\infty} \mathcal{H}_m(B) \otimes 1,$$

where $\mathcal{H}_m(B)$ is the self-adjoint operator acting in $L^2(I, r dr)$, given by

$$\mathcal{H}_m(B) = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \left(\frac{m}{r} - Ba(r)\right)^2,$$

with Neumann boundary conditions at the endpoints of I . The quadratic form corresponding to $\mathcal{H}_m(B)$ is given by

$$\mathfrak{q}_m[\psi] = \int_I \left[|\psi'(r)|^2 + \left(\frac{m}{r} - Ba(r)\right)^2 |\psi(r)|^2 \right] r \, dr. \quad (2.5)$$

It holds that

$$\lambda_{1, \mathcal{H}(B)} = \inf_{m \in \mathbb{Z}} \lambda_{1, \mathcal{H}_m(B)}.$$

3. THE ANALYSIS OF THE ANNULUS

3.1. Introduction. In this section we will let

$$\beta(x) = 1 \quad \text{and} \quad \Omega = \{x \in \mathbb{R}^2 \mid R_i < |x| < R_o\}.$$

We aim to prove Theorem 1.1.

3.2. The linear result. We first notice the non-monotonicity of the function $B \mapsto \lambda_{1,\mathcal{H}(B)}$.

Theorem 3.1. *Let $R_i = 1$ and $1 < R_o < \sqrt{2}$. Then the operator $\mathcal{H}(B)$ in the annulus Ω satisfies*

$$\left. \frac{d}{dB} \lambda_{1,\mathcal{H}(B)} \right|_{B=1} < 0.$$

In particular, the function $B \mapsto \lambda_{1,\mathcal{H}(B)}$ is monotonically decreasing around $B = 1$.

One might suspect that some properties of $\mathcal{H}(B)$ are carried over to some model problem on the circle, as $R_o \searrow R_i$. Let $\mathcal{A}(B)$ be the self-adjoint operator

$$\mathcal{A}(B) = \left(\frac{i}{R_i} \frac{d}{d\theta} - \frac{BR_i}{2} \right)^2$$

in $L^2((0, 2\pi))$ with periodic boundary conditions. Its spectrum is easily seen to consist of eigenvalues $\left\{ \left(\frac{m}{R_i} - \frac{BR_i}{2} \right)^2 \right\}_{m \in \mathbb{Z}}$. In particular

$$\lambda_{1,\mathcal{A}(B)} = \min_{m \in \mathbb{Z}} \left(\frac{m}{R_i} - \frac{BR_i}{2} \right)^2.$$

Our next theorem states that $\lambda_{1,\mathcal{H}(B)}$ will tend to $\lambda_{1,\mathcal{A}(B)}$ as $R_o \searrow R_i$.

Theorem 3.2. *Let $B > 0$. Then*

$$\lim_{R_o \searrow R_i} \lambda_{1,\mathcal{H}(B)} = \lambda_{1,\mathcal{A}(B)} = \min_{m \in \mathbb{Z}} \left(\frac{m}{R_i} - \frac{BR_i}{2} \right)^2.$$

Remark 3.3. As a direct consequence of Theorem 3.2 it is possible to find an annulus such that the function $B \mapsto \lambda_{1,\mathcal{H}(B)}$ is monotonically increasing and decreasing alternatively as many times as desired.

Remark 3.4. Another direct consequence of Theorem 3.2 is that, although the diamagnetic inequality tells us that $\lambda_{1,\mathcal{H}(B)} > \lambda_{1,\mathcal{H}(0)} = 0$ for all $B > 0$ we can actually get $\lambda_{1,\mathcal{H}(B)}$ to be arbitrary close to zero if $B = 2m$, $m = 1, 2, \dots$ by choosing R_o close enough to R_i .

Remark 3.5. Theorem 3.2 can easily be extended to thin cylinders in three dimensions, since the third variable then separates.

3.3. Nonmonotonicity in the annulus. In this section we will prove the spectral results Theorem 3.1 and Theorem 3.2. We will work in polar coordinates.

Proof of Theorem 3.1. We recall that here $R_i = 1$. Let

$$p_{m,B}(r) = \left(\frac{m}{r} - \frac{Br}{2} \right)^2$$

denote the potential in the quadratic form \mathfrak{q}_m in (2.5).

We start by showing that if $R_o > 1$ and $m \in \mathbb{Z} \setminus \{1\}$ then

$$\lambda_{1,\mathcal{H}_m(1)} > \lambda_{1,\mathcal{H}_1(1)}. \quad (3.1)$$

The function $f(r) = p_{m,1}(r) - p_{1,1}(r)$ is positive for $r > 1$. Indeed, $f(r) = 1 - m + (m^2 - 1)/r^2$. If $m \notin \{0, 1\}$ then f is decreasing, and $f(r) \geq f(1) = m^2 - m > 0$. If $m = 0$ then $f(r) = 1 - 1/r^2$ which is clearly positive for all $r > 1$. The inequality (3.1) follows by a comparison of quadratic forms.

Next, we show that if $1 < R_o < \sqrt{2m/B}$ then

$$\frac{d}{dB} \lambda_{1,\mathcal{H}_m(B)} < 0. \quad (3.2)$$

By perturbation theory it holds that

$$\frac{d}{dB} \lambda_{1, \mathcal{H}_m(B)} = \int_1^{R_o} \left(\frac{Br^2}{2} - m \right) u(r)^2 r dr, \quad (3.3)$$

where u denotes the eigenfunction corresponding to $\lambda_{1, \mathcal{H}_m(B)}$. Moreover the factor $\left(\frac{Br^2}{2} - m \right)$ is negative for all $1 < r < R_o$ if $R_o < \sqrt{2m/B}$. Inserting this into (3.3) gives (3.2)

It is now easy to finish the proof of Theorem 3.1. Let $1 < R_o < \sqrt{2}$. Inequality (3.1) and analytic perturbation theory imply that

$$\lambda_{1, \mathcal{H}(B)} = \lambda_{1, \mathcal{H}_1(B)}$$

for B in a neighborhood of 1. Since, by (3.2), it holds that the derivative of $\lambda_{1, \mathcal{H}_1(B)}$ is negative at $B = 1$ the same is true for the derivative of $\lambda_{1, \mathcal{H}(B)}$. By continuity of the derivative this holds in a neighborhood of $B = 1$. In particular we conclude that the function $B \mapsto \lambda_{1, \mathcal{H}(B)}$ is strictly decreasing for these values of B . \square

Proof of Theorem 3.2. Since

$$\lambda_{1, \mathcal{H}(B)} = \inf_m \lambda_{1, \mathcal{H}_m(B)},$$

Theorem 3.2 is a direct consequence of the fact that, for $m \in \mathbb{Z}$ and $B \geq 0$,

$$\lim_{R_o \searrow R_i} \lambda_{1, \mathcal{H}_m(B)} = \left(\frac{m}{R_i} - \frac{BR_i}{2} \right)^2. \quad (3.4)$$

To get an upper bound we use a trial state. In fact, we use the simplest possible one. Let $u = \sqrt{2/(R_o^2 - R_i^2)}$. Then $\|u\|_{L^2((R_i, R_o), r dr)} = 1$. A simple calculation shows that

$$\begin{aligned} \lim_{R_o \searrow R_i} \mathfrak{q}_m[u] &= \lim_{R_o \searrow R_i} \left(\frac{2m^2}{R_o + R_i} \frac{\log R_o - \log R_i}{R_o - R_i} - Bm + \frac{B^2}{8} (R_i^2 + R_o^2) \right) \\ &= \left(\frac{m}{R_i} - \frac{BR_i}{2} \right)^2. \end{aligned}$$

Hence $\lim_{R_o \searrow R_i} \lambda_{1, \mathcal{H}_m(B)} \leq \left(\frac{m}{R_i} - \frac{BR_i}{2} \right)^2$.

The lower bound is obtained by using the potential $p_{m, B}(r)$. Let u be a normalized eigenfunction corresponding to $\lambda_{1, \mathcal{H}_m(B)}$. then

$$\lambda_{1, \mathcal{H}_m(B)} = \mathfrak{q}_m[u] \geq \int_{R_i}^{R_o} \left(\frac{Br}{2} - \frac{m}{r} \right)^2 |u|^2 r dr \geq \min_{R_i \leq r \leq R_o} \left(\frac{Br}{2} - \frac{m}{r} \right)^2.$$

Since $\min_{R_i \leq r \leq R_o} \left(\frac{Br}{2} - \frac{m}{r} \right)^2 \rightarrow \left(\frac{m}{R_i} - \frac{BR_i}{2} \right)^2$ as $R_o \searrow R_i$ we conclude that

$$\lim_{R_o \searrow R_i} \lambda_{1, \mathcal{H}_m(B)} \geq \left(\frac{m}{R_i} - \frac{BR_i}{2} \right)^2.$$

This completes the proof of (3.4), and thus finishes the proof of Theorem 3.2. \square

3.4. Application to the Ginzburg–Landau functional. In this section we prove Theorem 1.1. We recall the reader that D below denotes the disc with radius R_o , centered at the origin. We need the following lemma, and refer to [8] for its proof.

Lemma 3.6. *Let R_i be fixed and let $R_i \leq R_o \leq 2$. There exists a constant $\widehat{C} > 0$ (independent of R_o) such that for all $\mathbf{a} \in H_{\text{div}}^1(D)$ we have*

$$\|\mathbf{a}\|_{L^2(D)} \leq \widehat{C} \|\text{curl } \mathbf{a}\|_{L^2(D)}.$$

Combining this with the Sobolev embedding we get the existence of a constant \widehat{C}_0 (independent of $R_o \in [R_i, 2]$) such that for all $\mathbf{a} \in H_{\text{div}}^1(D)$

$$\|\mathbf{a}\|_{L^4(D)} \leq \widehat{C}_0 \|\text{curl } \mathbf{a}\|_{L^2(D)}. \quad (3.5)$$

Proof of Theorem 1.1. Given $0 < \varepsilon < 1$, the Cauchy inequality implies that

$$|(i\nabla + \kappa\sigma\mathbf{A})\psi|^2 \geq (1 - \varepsilon)|(i\nabla + \kappa\sigma\mathbf{F})\psi|^2 - \varepsilon^{-1}(\kappa\sigma)^2|\mathbf{A} - \mathbf{F}|^2|\psi|^2,$$

and so

$$\begin{aligned} \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) &\geq \int_{\Omega} (1 - \varepsilon)|(i\nabla + \kappa\sigma\mathbf{F})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 dx \\ &\quad - \varepsilon^{-1}(\kappa\sigma)^2 \int_{\Omega} |\mathbf{A} - \mathbf{F}|^2|\psi|^2 dx + (\kappa\sigma)^2 \int_D |\text{curl } \mathbf{A} - 1|^2 dx \\ &\geq ((1 - \varepsilon)\lambda_{1,\mathcal{H}(\kappa\sigma)} - \kappa^2) \|\psi\|_{L^2(\Omega)}^2 \\ &\quad - \varepsilon^{-1}(\kappa\sigma)^2 \|\mathbf{A} - \mathbf{F}\|_{L^4(D)}^2 \|\psi\|_{L^4(\Omega)}^2 + (\kappa\sigma)^2 \int_D |\text{curl } \mathbf{A} - 1|^2 dx \\ &\geq ((1 - \varepsilon)\lambda_{1,\mathcal{H}(\kappa\sigma)} - \kappa^2) \|\psi\|_{L^2(\Omega)}^2 \\ &\quad + (\kappa\sigma)^2 \left(1 - \widehat{C}_0^2 \varepsilon^{-1} \sqrt{\pi}(R_o^2 - R_i^2)^{1/2}\right) \int_D |\text{curl } \mathbf{A} - 1|^2 dx. \end{aligned}$$

Here we used (3.5) and $\|\psi\|_{\infty} \leq 1$ to get the last inequality.

If we choose $\varepsilon = (R_o - R_i)^{1/4}$, then we see that if $\lambda_{1,\mathcal{A}(\kappa\sigma)} > \kappa^2$, then for all R_o sufficiently close to R_i and all (ψ, \mathbf{A}) ,

$$\mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) \geq 0. \quad (3.6)$$

On the other hand, if $\lambda_{1,\mathcal{H}(B=\sigma\kappa)} < \kappa^2$, then we have (with $\mathbf{F} = 1/2(-x_2, x_1)$ and u the normalized eigenfunction corresponding to $\lambda_{1,\mathcal{H}(\sigma\kappa)}$)

$$\mathcal{G}_{\kappa,\sigma}(cu, \mathbf{F}) = c^2(\lambda_{1,\mathcal{H}(\sigma\kappa)} - \kappa^2) + c^4 \frac{\kappa^2}{2} \int_{\Omega} |u|^4 dx < 0 \quad (3.7)$$

for sufficiently small values of c .

Therefore, by the explicit spectrum of $\mathcal{A}(B)$ we can choose $\kappa_0 > 0$ and $B_0 < B_1 < B_2$ such that

$$\lambda_{1,\mathcal{A}(B_j)} < \kappa_0^2, \quad j = 0, 2, \quad \lambda_{1,\mathcal{A}(B_1)} > \kappa_0^2.$$

Define $\sigma_j := B_j/\kappa_0$. By the convergence of the spectrum given in Theorem 3.2 and (3.7) we find the existence of $\widetilde{R} > R_i$ such that $\mathcal{G}_{\kappa_0,\sigma_j}$ has a non-trivial minimizer for all $R_i < R_o \leq \widetilde{R}$ and $j \in \{0, 2\}$. On the other hand, it follows from (3.6) that the minimizer of $\mathcal{G}_{\kappa_0,\sigma_1}$ is trivial for all $R_o > R_i$ sufficiently close to R_i .

We conclude the existence of $R_o > R_i$ such that there exist non-trivial minimizers when $\sigma = \sigma_0$ and $\sigma = \sigma_2$ but not when $\sigma = \sigma_1$. Since $\sigma_0 < \sigma_1 < \sigma_2$ it is clear that $\mathcal{N}(\kappa_0)$ is not an interval. \square

4. THE CASE OF THE COMPLEMENT OF THE DISC

4.1. Introduction. In this section we consider the case $\Omega = \{x \in \mathbb{R}^2 : |x| > 1\}$ and assume that the magnetic field satisfies Assumption 1.5 with $\delta > 0$. Our aim is to prove Theorem 1.10.

4.2. Localization estimate. Before continuing we give an Agmon estimate for the lowest eigenfunction.

Proposition 4.1. *Assume that β satisfies Assumption 1.5 with $\delta > 0$. Let $t \in (0, 1)$. Then there exist positive constants C , a and B_0 such that if $B > B_0$, and if ψ is an eigenfunction of $\mathcal{H}(B)$ corresponding to an eigenvalue $\lambda \leq t\delta B$. then*

$$\int_{\{|x|>1\}} \exp(aB^{1/2}||x| - 1|)(|\psi|^2 + B^{-1}|(-i\nabla + B\mathbf{F})\psi|^2) dx \leq C \int_{\{|x|>1\}} |\psi|^2 dx.$$

Theorem 8.2.4 in [8] gives the same estimate with the restriction that the domain should be bounded. However, since we give a similar Agmon estimate in Section 7 with proof we omit the proof here.

4.3. A detailed expansion. We recall that the quadratic form after decomposition is given by (with $a(r)$ from (2.1))

$$\mathfrak{q}_m[u] = \int_1^{+\infty} \left(|u'(r)|^2 + \left(\frac{m}{r} - Ba(r) \right)^2 |u(r)|^2 \right) r dr.$$

Notice that at $r = 1$ the potential takes the value

$$\left(\frac{m}{r} - Ba(r) \right)^2 \Big|_{r=1} = (m - \Phi B)^2.$$

This suggests that we will find the lowest energy for $m \approx \Phi B$. That this is the case is the content of the following Lemma.

Lemma 4.2. *Let $t \in (0, 1)$. Suppose $\psi = u_m e^{-im\theta}$ is an eigenfunction of $\mathcal{H}(B)$ with eigenvalue $\lambda \leq t\delta B$. Then*

$$m = \Phi B + \mathcal{O}(B^{1/2}).$$

Proof. We neglect the kinetic energy in the expression for \mathfrak{q}_m . Recall the calculation (2.2). For $1 < r < 2$, we get

$$\left| \int_0^{r-1} (1+s)\tilde{\beta}(1+s) ds \right| \leq C(r-1), \quad (4.1)$$

so, estimating the quadratic form with the potential, combining (4.1) and (2.3), and using Proposition 4.1, we get

$$\begin{aligned} \mathfrak{q}_m[u_m] &\geq \int_1^2 \frac{1}{r^2} (m - Bra(r))^2 |u_m(r)|^2 r dr \\ &\geq \int_1^2 \frac{1}{r^2} \left[\frac{1}{2} (m - \Phi B)^2 - (CB)^2 (r-1)^2 \right] |u_m(r)|^2 r dr \\ &\geq \frac{1}{8} (m - \Phi B)^2 (1 + \mathcal{O}(B^{-\infty})) - \tilde{C}B, \end{aligned} \quad (4.2)$$

from which the lemma follows. \square

Lemma 4.3. *Let $t \in (0, 1)$. There exists $B_0 > 0$ such that if $m \in \mathbb{Z}$ and $B \geq B_0$, then $\mathcal{H}_m(B)$ admits at most one eigenvalue below $t\delta B$.*

Proof. Fix \tilde{t} with $t < \tilde{t} < 1$. By the lower bound (4.2), we see that there exist $B_0, C_0 > 0$ such that if $|m - \Phi B| \geq C_0 B^{1/2}$, then $\mathfrak{q}_m \geq \tilde{t}\delta B$.

So we will restrict attention to m 's such that $m = \Phi B + \Delta m$, with $|\Delta m| \leq C_0 B^{1/2}$. Suppose, to get a contradiction, that u_1, u_2 are eigenfunctions of \mathfrak{q}_m corresponding to eigenvalues below $t\delta B$.

We write

$$\left(\frac{m}{r} - Ba(r) \right)^2 = \frac{1}{r^2} (m - Bra(r))^2,$$

with

$$ra(r) = \Phi + \delta(r-1) + \mathcal{O}((r-1)^2),$$

as $r \rightarrow 1$. So

$$|m - Bra(r)| \geq |m - \Phi B - B\delta(r-1)| + \mathcal{O}(B(r-1)^2).$$

Using the Agmon estimates, this yields the following bound on normalized functions v in $\text{span}\{u_1, u_2\}$.

$$\begin{aligned} \mathfrak{q}_m[v] &\geq \int_1^\infty \left(|v'(r)|^2 + \frac{1}{r^2} (\Delta m - B\delta(r-1))^2 |v(r)|^2 \right) r dr + \mathcal{O}(B^{1/2}) \\ &= \tilde{\mathfrak{q}}_m[v] + \mathcal{O}(B^{1/2}), \end{aligned} \quad (4.3)$$

with

$$\tilde{\mathfrak{q}}_m[v] = \int_1^\infty |v'(r)|^2 + (\Delta m - B\delta(r-1))^2 |v(r)|^2 dr. \quad (4.4)$$

By translation and scaling $\tilde{\mathfrak{q}}_m$ is unitarily equivalent to (the quadratic form of) a de Gennes operator (see Appendix A) and therefore has spectrum given by

$$B\delta \{ \lambda_{j, \mathcal{H}_{\text{dG}}}(\Delta m / (\delta B)^{1/2}) \}_{j=1}^{+\infty}$$

Only the first of these $\lambda_{j, \mathcal{H}_{\text{dG}}}$ —counted with multiplicity—is below 1 (for some values of $\Delta m / (\delta B)^{1/2}$), so we reach a contradiction if we have a subspace of dimension 2 on which the quadratic form is small. \square

Lemma 4.4. *Let $M > 0$. Suppose $\mathcal{H}_m(B)$ admits an eigenvalue below $\Theta_0 \delta B + MB^{1/2}$. Then there exists a constant $C > 0$ such that*

$$|m - (\Phi B + \xi_0(\delta B)^{1/2})| \leq CB^{1/4}. \quad (4.5)$$

Proof. By Lemma 4.2, $|m - \Phi B| = \mathcal{O}(B^{1/2})$. Assuming that u is the eigenfunction corresponding to the unique (by Lemma 4.3) eigenvalue λ below $\Theta_0 \delta B + MB^{1/2}$ we can use the estimate in (4.3) to find that

$$\mathfrak{q}_m[u] \geq \tilde{\mathfrak{q}}_m[u] + \mathcal{O}(B^{1/2}),$$

with $\tilde{\mathfrak{q}}_m$ as in (4.4). Implementing the change of variable $r = 1 + (\delta B)^{-1/2} \rho$, we get (here we write $v(\rho) = (\delta B)^{-1/4} u(1 + (\delta B)^{-1/2} \rho)$)

$$\tilde{\mathfrak{q}}_m[v] = \delta B \int_0^{+\infty} |v'(\rho)|^2 + \left(\rho - \xi_0 + \xi_0 - \frac{m - \Phi B}{(\delta B)^{1/2}} \right)^2 |v|^2 d\rho.$$

We recognize this as the quadratic form for the de Gennes operator (see Appendix A). By noticing that the first eigenvalue $\lambda_{1, \mathcal{H}_{\text{dG}}}(\xi)$ has a quadratic minimum Θ_0 at ξ_0 (and using the bound on $(m - \Phi B)/(\delta B)^{1/2}$) we find that there exists a positive constant C_0 such that

$$\tilde{\mathfrak{q}}_m[v] \geq \left[\Theta_0 \delta B + C_0 \delta B \left(\xi_0 - \frac{m - \Phi B}{(\delta B)^{1/2}} \right)^2 \right] \|v\|^2.$$

The second term above is bounded by some constant times $B^{1/2}$ according to the assumption. This in turn gives the existence of a positive constant C such that (4.5) holds. \square

In the remainder of this section we will always restrict our attention to m 's satisfying the conclusion of Lemma 4.4.

The strategy of the rest of the proof is as follows. We will construct an explicit trial state for the operator $\mathfrak{h} = \frac{1}{B} \mathcal{H}_m(B)$ (here we suppress the dependence on m and B for the simplicity of notation). This trial state will be constructed as the first terms of a formal expansion. By taking only finitely many terms (for our purposes

3 terms suffice) and performing a localization one gets a well-defined trial state. In terms of the objects calculated below our explicit trial state will be as follows. Let

$$v(\rho) = v_0 + B^{-1/2}v_1 + B^{-1}v_2, \quad \lambda = \lambda_0 + B^{-1/2}\lambda_1 + B^{-1}\lambda_2.$$

Let furthermore, $\chi \in C_0^\infty(\mathbb{R})$, with $\chi(0) = 1$, and define (with suitable ε , say $\varepsilon = (100)^{-1}$)

$$\tilde{v}(r) = (\delta B)^{1/4} \chi(B^{1/2-\varepsilon}(r-1)) v((\delta B)^{1/2}(r-1)). \quad (4.6)$$

Then $\|\tilde{v}\|_{L^2} = 1 + \mathcal{O}(B^{-1/2})$ and

$$\|(\mathfrak{h} - \lambda)\tilde{v}\| = \mathcal{O}(B^{-3/2}).$$

By self-adjointness of \mathfrak{h} we get that $\text{dist}(\lambda, \sigma(\mathfrak{h})) = \mathcal{O}(B^{-3/2})$. Since we by Lemma 4.3 know that \mathfrak{h} has at most one eigenvalue near $\lambda_0 = \delta\Theta_0$, we can conclude that λ gives the first terms of the asymptotic expansion of that lowest eigenvalue of \mathfrak{h} .

We proceed to the termwise construction of the trial state. Since (by Proposition 4.1) we have localization around $r = 1$, we implement unitarily the change of variables

$$\rho = (\delta B)^{1/2}(r-1), \quad r = 1 + (\delta B)^{-1/2}\rho.$$

Here, the δ is included for convenience. Then

$$\text{Bra}(r) = \Phi B + (\delta B)^{1/2}\rho + \frac{1}{2}\rho^2 + \frac{k}{6\delta^{3/2}}B^{-1/2}\rho^3 + \mathcal{O}(B^{-1}).$$

Here the estimate on the remainder should be understood in the following sense: We will only act with our operator on the function \tilde{v} from (4.6) which is localized near $r = 1$ on the scale $B^{-1/2}$. So we may consider ρ as a quantity of order 1.

By Lemma 4.4 the constant term $m - \Phi B$ vanishes to leading order. For reasons of expositions we will write

$$m = \Phi B + \mu_1 B^{1/2} + \mu_2,$$

and not insert the choice $\mu_1 = \xi_0 \delta^{1/2}$ until later. Recall that $\mu_2 B^{-1/4}$ is bounded. Integrating by parts, we find (with $v(\rho) = (\delta B)^{-1/4}u(1 + (\delta B)^{-1/2}\rho)$)

$$\begin{aligned} & \frac{1}{B} \int_1^{+\infty} \left| \frac{du}{dr} \right|^2 r dr \\ &= \delta \int_0^{+\infty} \bar{v} \left(-\frac{d^2 v}{d\rho^2} - (\delta B)^{-1/2} (1 + (\delta B)^{-1/2}\rho)^{-1} \frac{dv}{d\rho} \right) (1 + (\delta B)^{-1/2}\rho) d\rho. \end{aligned}$$

We expand our operator \mathfrak{h} as

$$\mathfrak{h} = \mathfrak{h}_0 + B^{-1/2}\mathfrak{h}_1 + B^{-1}\mathfrak{h}_2 + \dots$$

and obtain

$$\begin{aligned} \mathfrak{h}_0 &= \delta \left(-\frac{d^2}{d\rho^2} + (\rho - \mu_1/\delta^{1/2})^2 \right), \\ \mathfrak{h}_1 &= -\delta^{1/2} \frac{d}{d\rho} - 2\mu_2 \delta^{1/2} (\rho - \mu_1/\delta^{1/2}) - \frac{2\mu_1^2}{\delta^{1/2}} \rho + 3\mu_1 \rho^2 - \delta^{1/2} \rho^3, \\ \mathfrak{h}_2 &= \rho \frac{d}{d\rho} + \mu_2^2 - \frac{4\mu_1 \mu_2}{\delta^{1/2}} \rho + 3\mu_2 \rho^2 + \frac{3\mu_1^2}{\delta} \rho^2 - \frac{k\mu_1}{3\delta^{3/2}} \rho^3 - \frac{4\mu_1}{\delta^{1/2}} \rho^3 + \frac{k}{3\delta} \rho^4 + \frac{5}{4} \rho^4. \end{aligned} \quad (4.7)$$

We make the Ansatz

$$v = \sum_{j=0}^{+\infty} v_j B^{-j/2}, \quad \lambda = \sum_{j=0}^{+\infty} \lambda_j B^{-j/2}.$$

Equating order by order in the relation $(\mathfrak{h} - \lambda)v = 0$ gives:

Order B^0 : To leading order we find

$$\mathfrak{h}_0 v_0 = \lambda_0 v_0,$$

which is the eigenvalue problem for the de Gennes operator discussed in Appendix A. The optimal eigenvalue $\lambda_0 = \delta\Theta_0$ is attained for $v_0 = \varphi_{\xi_0}$ and $\mu_1 = \delta^{1/2}\xi_0$.

Order $B^{-1/2}$: Here we get

$$(\mathfrak{h}_0 - \lambda_0)v_1 = (\lambda_1 - \mathfrak{h}_1)v_0.$$

By taking scalar product (with measure $d\rho$), we find

$$0 = \langle v_0, (\mathfrak{h}_0 - \lambda_0)v_1 \rangle = \lambda_1 - \langle v_0, \mathfrak{h}_1 v_0 \rangle.$$

Via the formulas (A.1)–(A.2) we find

$$\begin{aligned} \lambda_1 &= \langle \varphi_{\xi_0}, \mathfrak{h}_1 \varphi_{\xi_0} \rangle \\ &= \left\langle \varphi_{\xi_0}, \left(-\delta^{1/2} \frac{d}{d\rho} - 2\mu_2 \delta^{1/2} (\rho - \mu_1 / \delta^{1/2}) - \frac{2\mu_1^2}{\delta^{1/2}} \rho + 3\mu_1 \rho^2 - \delta^{1/2} \rho^3 \right) \varphi_{\xi_0} \right\rangle \\ &= -\delta^{1/2} \langle \varphi_{\xi_0}, \varphi'_{\xi_0} \rangle - 2\mu_2 \delta^{1/2} \langle \varphi_{\xi_0}, (\rho - \xi_0) \varphi_{\xi_0} \rangle - 2\xi_0^2 \delta^{1/2} \langle \varphi_{\xi_0}, \rho \varphi_{\xi_0} \rangle \\ &\quad + 3\xi_0 \delta^{1/2} \langle \varphi_{\xi_0}, \rho^2 \varphi_{\xi_0} \rangle - \delta^{1/2} \langle \varphi_{\xi_0}, \rho^3 \varphi_{\xi_0} \rangle \\ &= \frac{1}{3} \varphi_{\xi_0}(0)^2 \delta^{1/2}. \end{aligned}$$

In particular λ_1 is independent of μ_2 . Moreover, since we can choose $v_1 \perp v_0$, we can let v_1 be the regularized resolvent $(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1}$ of $-\mathfrak{h}_1 v_0$. This regularized resolvent is defined as the inverse of the operator $(\mathfrak{h}_0 - \lambda_0)$ restricted to the space $\{v_0\}^\perp$. So we have,

$$v_1 = -(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1} [\mathfrak{h}_1 v_0]. \quad (4.8)$$

Order B^{-1} : We get

$$(\mathfrak{h}_0 - \lambda_0)v_2 = (\lambda_2 - \mathfrak{h}_2)v_0 + (\lambda_1 - \mathfrak{h}_1)v_1. \quad (4.9)$$

Taking scalar product with v_0 again gives

$$\lambda_2 = \langle v_0, \mathfrak{h}_2 v_0 \rangle + \langle v_0, (\mathfrak{h}_1 - \lambda_1)v_1 \rangle.$$

We will not calculate this expression in all detail. We are only interested in the dependence on μ_2 . An inspection gives that it will be a polynomial of degree two. We will calculate the coefficient in front of μ_2^2 to see that it is positive so that λ_2 has a unique minimum with respect to μ_2 .

The term $\langle v_0, \mathfrak{h}_2 v_0 \rangle$ is easily calculated since \mathfrak{h}_2 contains one μ_2^2 only.

For the term $\langle v_0, (\mathfrak{h}_1 - \lambda_1)v_1 \rangle$ we find one μ_2 in \mathfrak{h}_1 and therefore also one in v_1 . The coefficient in front of μ_2 in that term becomes

$$\begin{aligned} &\left\langle v_0, -2\delta^{1/2}(\rho - \xi_0)(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1} [2\delta^{1/2}(\rho - \xi_0)v_0] \right\rangle \\ &= -4 \left\langle (\rho - \xi_0) \varphi_{\xi_0}, (\mathcal{H}_{\text{dG}}(\xi_0) - \Theta_0)_{\text{reg}}^{-1} [(\rho - \xi_0) \varphi_{\xi_0}] \right\rangle. \end{aligned}$$

So, the coefficient in front of μ_2^2 in λ_2 will be (see (A.3))

$$1 - 4 \left\langle (\rho - \xi_0) \varphi_{\xi_0}, (\mathcal{H}_{\text{dG}}(\xi) - \Theta_0)_{\text{reg}}^{-1} [(\rho - \xi_0) \varphi_{\xi_0}] \right\rangle = \xi_0 \varphi_{\xi_0}(0)^2 > 0.$$

This means that we can write

$$\lambda_2 = \xi_0 \varphi_{\xi_0}(0)^2 \left((\mu_2 - C_0^{\text{ext}})^2 + C_1^{\text{ext}} \right), \quad (4.10)$$

where C_0^{ext} and C_1^{ext} depend only on k , δ , ξ_0 and $\varphi_{\xi_0}(0)$ (but not on Φ).

We summarize these findings in a Lemma.

Lemma 4.5. *Suppose*

$$m = \Phi B + \xi_0 (\delta B)^{1/2} + \mu_2,$$

with $\mu_2 = \mathcal{O}(B^{1/4})$. Then

$$\begin{aligned} \lambda_{1, \mathcal{H}_m(B)} &= \Theta_0 \delta B + \frac{1}{3} \varphi_{\xi_0}(0)^2 (\delta B)^{1/2} + \xi_0 \varphi_{\xi_0}(0)^2 ((\mu_2 - C_0^{\text{ext}})^2 + C_1^{\text{ext}}) \\ &\quad + \mathcal{O}((1 + \mu_2^3) B^{-1/2}). \end{aligned}$$

Proof. We have to control the asymptotic expansion in μ_2 subject to the bound $|\mu_2| \leq CB^{1/4}$. Define

$$\lambda^{\text{app}} = \lambda_0 + \lambda_1 B^{-1/2} + \lambda_2 B^{-1},$$

with λ_0, λ_1 being the constants from above and λ_2 being the quadratic function of μ_2 from (4.10). We also define an approximate eigenfunction by

$$v = v_0 + B^{-1/2} v_1 + B^{-1} v_2,$$

with $v_0 = \varphi_{\xi_0}$, v_1 given by (4.8) and v_2 being given by solving (4.9), i.e.

$$v_2 = (\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1} [(\lambda_2 - \mathfrak{h}_2) v_0 + (\lambda_1 - \mathfrak{h}_1) v_1].$$

Notice from the explicit form of the operators that v_1 depends linearly on μ_2 and v_2 depends quadratically, so v is normalized to leading order. Also, by the mapping properties of $(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1}$ each v_i is a smooth, rapidly decreasing function (see Lemma 3.2.9 in [8]).

We can now estimate as follows

$$\begin{aligned} \|(\mathfrak{h} - \lambda^{\text{app}})v\| &\leq \|(\mathfrak{h}_0 + B^{-1/2} \mathfrak{h}_1 + B^{-1} \mathfrak{h}_2 - \lambda^{\text{app}})v\| \\ &\quad + \|(\mathfrak{h} - [\mathfrak{h}_0 + B^{-1/2} \mathfrak{h}_1 + B^{-1} \mathfrak{h}_2])v\|. \end{aligned}$$

By the decay properties of v , the last term is bounded by $C(1 + \mu_2^2)B^{-3/2}$. Our choice of v gives that the first term is equal to

$$\|B^{-3/2}[(\mathfrak{h}_1 - \lambda_1)v_2 + (\mathfrak{h}_2 - \lambda_2)v_1] + B^{-2}(\mathfrak{h}_2 - \lambda_2)v_2\|,$$

which is easily seen to be bounded by $\mathcal{O}(B^{-3/2}(1 + |\mu_2|^3))$. \square

Proof of Theorem 1.10. Using Lemma 4.4, Theorem 1.10 follows from Lemma 4.5 by the following argument. Notice that the positive quadratic term in $(\mu_2 - C_0^{\text{ext}})$ dominates the error term $\mu_2^3 B^{-1/2}$ unless μ_2 is bounded in which case the dependence on μ_2 in the error term disappears. This finishes the proof of Theorem 1.10. \square

5. THE CASE OF THE DISC

In this section we will indicate a similar calculation of the ground state eigenvalue in the case of the unit disc, thereby proving Theorem 1.12, i.e. we work on $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ and for a magnetic field satisfying Assumption 1.5.

We mainly give the results of the calculations referring to the exterior case for details. We will have exponential localization estimate like the one of Proposition 4.1 (with domain of integration being $\{|x| < 1\}$, of course). Therefore, also the rough ‘localization’ of the relevant angular momenta—Lemma 4.2—will hold in this case as well. So we can proceed to make a change of variable to the region near (on the scale $(\delta B)^{-1/2}$ as before) the boundary.

The leading order terms in the expansion of the operator become very similar to the case of the exterior of the disc:

$$\begin{aligned}
\mathfrak{h}_0 &= \delta \left(-\frac{d^2}{d\rho^2} + (\rho + \mu_1/\delta^{1/2})^2 \right), \\
\mathfrak{h}_1 &= \delta^{1/2} \frac{d}{d\rho} + 2\mu_2\delta^{1/2}(\rho + \mu_1/\delta^{1/2}) + \frac{2\mu_1^2}{\delta^{1/2}}\rho + 3\mu_1\rho^2 + \delta^{1/2}\rho^3, \\
\mathfrak{h}_2 &= \rho \frac{d}{d\rho} + \mu_2^2 + \frac{4\mu_1\mu_2}{\delta^{1/2}}\rho + 3\mu_2\rho^2 + \frac{3\mu_1^2}{\delta}\rho^2 + \frac{k\mu_1}{3\delta^{3/2}}\rho^3 + \frac{4\mu_1\rho^3}{\delta^{1/2}} + \frac{k}{3\delta}\rho^4 + \frac{5}{4}\rho^4.
\end{aligned}$$

The same calculations (using the same Ansatz) as in the previous section show that (with $\mu_1 = -\xi_0/\delta^{1/2}$)

$$\lambda_0 = \Theta_0, \quad \lambda_1 = -\frac{1}{3}\varphi_{\xi_0}(0)^2\delta^{1/2}, \quad \lambda_2 = \xi_0\varphi_{\xi_0}(0)^2\left((\mu_2 - C_0^{\text{int}})^2 + C_1^{\text{int}}\right),$$

for some constants C_0^{int} and C_1^{int} , depending only on the spectral parameters and δ .

Thus, Theorem 1.12 follows from calculations/arguments completely analogous to the ones in Section 4 and we omit the details.

6. (NON)-MONOTONICITY IN THE DISC AND ITS COMPLEMENT

Using the results of Theorem 1.10 and 1.12 it is now easy to prove Theorem 1.8.

Proof of Theorem 1.8. We only consider the case of the disc, the complement of the disc being similar (using Theorem 1.10 instead of Theorem 1.12).

Assume first that

$$\Phi > \frac{\Theta_0}{\xi_0\varphi_{\xi_0}(0)^2}\delta.$$

Denote by f the function

$$f(B) = \Phi B - \xi_0(\delta B)^{1/2} + C_0^{\text{int}}.$$

Notice that $B \mapsto f(B)$ is increasing for all large values of B . Choose a sequence $\{B_1^{(n)}\}$ such that $f(B_1^{(n)}) = n + 1/2$, i.e. is a half-integer. Let $\varepsilon \in (0, \frac{1}{2\Phi})$. Choose $B_2^{(n)} = B_1^{(n)} + \varepsilon$. Then, for all sufficiently large n , $n + 1/2 < f(B_2^{(n)}) < n + 1$. So $\Delta_{B_1^{(n)}}^{\text{int}} = 1/2$ and

$$\lim_{n \rightarrow +\infty} \Delta_{B_2^{(n)}}^{\text{int}} = \lim_{n \rightarrow +\infty} (n + 1 - f(B_2^{(n)})) = \frac{1}{2} - \Phi\varepsilon.$$

So we get from the eigenvalue asymptotics that

$$\begin{aligned}
\lambda_{1, \mathcal{H}(B_2^{(n)})} - \lambda_{1, \mathcal{H}(B_1^{(n)})} &= \Theta_0\delta(B_2^{(n)} - B_1^{(n)}) - \frac{1}{3}\varphi_{\xi_0}(0)^2\delta^{1/2}[(B_2^{(n)})^{1/2} - (B_1^{(n)})^{1/2}] \\
&\quad + \xi_0\varphi_{\xi_0}(0)^2[(1/2 - \Phi\varepsilon)^2 - 1/4] + o(1) \\
&= \Theta_0\delta\varepsilon - \xi_0\varphi_{\xi_0}(0)^2[\Phi\varepsilon - \Phi^2\varepsilon^2] + o(1),
\end{aligned}$$

which is negative for small ε (and for all sufficiently large n) since $\Phi > \frac{\Theta_0}{\xi_0\varphi_{\xi_0}(0)^2}\delta$ by assumption.

Suppose now that

$$\Phi < \frac{\Theta_0}{\xi_0\varphi_{\xi_0}(0)^2}\delta. \tag{6.1}$$

We restrict attention to the interval near infinity on which $f(B)$ is increasing. Here we can calculate the right-hand derivative

$$\frac{d}{dB+} (\Delta_B^{\text{int}})^2 = \begin{cases} 2\Delta_B^{\text{int}} f'(B), & \text{if } f(B) \in \mathbb{Z} + [0, 1/2), \\ -2\Delta_B^{\text{int}} f'(B), & \text{if } f(B) \in \mathbb{Z} + [1/2, 1). \end{cases}$$

So we see that for any $\eta > 0$ there exists $B_0 > 0$ such that for all $\varepsilon > 0$ and all $B > B_0$,

$$(\Delta_{B+\varepsilon}^{\text{int}})^2 - (\Delta_B^{\text{int}})^2 \geq -2 \int_B^{B+\varepsilon} \Delta_b^{\text{int}} f'(b) db \geq -(\Phi + \eta)\varepsilon. \quad (6.2)$$

We aim to prove monotonicity of $\lambda_{1,\mathcal{H}(B)}$, so it suffices to prove a positive lower bound on its right hand derivative $\frac{d}{dB_+} \lambda_{1,\mathcal{H}(B)}$, which exists by perturbation theory. Perturbation theory yields, for any $\varepsilon > 0$,

$$\begin{aligned} \frac{d}{dB_+} \lambda_{1,\mathcal{H}(B)} &= 2\Re\langle \psi_B, \mathbf{A} \cdot (-i\nabla + B\mathbf{A})\psi \rangle \\ &\geq \frac{\lambda_1(B+\varepsilon) - \lambda_1(B)}{\varepsilon} - \varepsilon \int_{\{|x|<1\}} \mathbf{A}^2 |\psi|^2 dx. \end{aligned}$$

Here we completed the square and used the variational characterization of the eigenvalue in order to get the inequality.

Since $\int_{\{|x|<1\}} \mathbf{A}^2 |\psi|^2 dx \leq K$, for some constant K independent of B , we can estimate, using the eigenvalue asymptotics and (6.2)

$$\liminf_{B \rightarrow +\infty} \frac{d}{dB_+} \lambda_{1,\mathcal{H}(B)} \geq \Theta_0 \delta - \xi_0 \varphi_{\xi_0}(0)^2 (\Phi + \eta) - \varepsilon K.$$

Since ε, η were arbitrary, we get that

$$\liminf_{B \rightarrow +\infty} \frac{d}{dB_+} \lambda_{1,\mathcal{H}(B)} \geq \Theta_0 \delta - \xi_0 \varphi_{\xi_0}(0)^2 \Phi.$$

In particular, $\lambda_{1,\mathcal{H}(B)}$ is monotone increasing for large value of B if (6.1) is satisfied. \square

7. THE CASE OF THE WHOLE PLANE WITH $\delta > 0$

7.1. Introduction. In this section we will consider the case $\Omega = \mathbb{R}^2$ and a magnetic field β satisfying Assumption 1.5 with $\delta > 0$. We aim to prove Theorem 1.14 for $\delta > 0$. This, however, follows directly once the asymptotic expansion in Theorem 1.15 is obtained, since then it follows that (see [8, Section 2.3])

$$\lim_{B \rightarrow +\infty} \frac{d}{dB} \lambda_{1,\mathcal{H}(B)} = \delta.$$

The proof of Theorem 1.15 follows the same idea as the proof of Theorem 1.10. We use a localization of the ground state to restrict the situation to certain values of the angular momentum. Then we show that if we find a trial state with low enough energy, it must be related to the ground state energy. Finally we expand our operator formally and construct a trial state that has the correct energy.

7.2. Agmon estimate for $\delta \geq 0$. We start with a localization estimate valid for $\delta \geq 0$. For $\delta = 0$ it gives the right length scale of the localization.

Proposition 7.1. *Suppose β satisfies Assumption 1.5 with $\delta \geq 0$. Let ψ be an eigenfunction of $\mathcal{H}(B)$ corresponding to an eigenvalue $\lambda \leq \delta B + \omega B^{1/2}$ for some $\omega > 0$. Then there exist positive constants C and B_0 such that*

$$\int_{\mathbb{R}^2} \exp(2B^{1/4}|1 - |x||) |\psi|^2 dx \leq C \int_{\mathbb{R}^2} |\psi|^2 dx \quad (7.1)$$

and

$$\int_{\mathbb{R}^2} \exp(2B^{1/4}|1 - |x||) |(-i\nabla + B\mathbf{F})\psi|^2 dx \leq C(\delta B + B^{1/2}) \int_{\mathbb{R}^2} |\psi|^2 dx \quad (7.2)$$

if $B > B_0$.

By the localization estimates of Proposition 7.1, the quadratic forms \mathfrak{q}_m are well approximated by harmonic oscillators, whose ground state eigenvalues are simple. This implies simplicity of the low-lying eigenvalues of $\mathcal{H}_m(B)$.

Lemma 7.2. *Let $\delta > 0$. Let $\omega > 0$. There exists $B_0 > 0$ such that if $m \in \mathbb{Z}$ and $B \geq B_0$, then $\mathcal{H}_m(B)$ admits at most one eigenvalue below $\delta B + \omega B^{1/2}$.*

The proof of Lemma 7.2 is similar to that of Lemma 4.3 and will be omitted.

Proof of Prop. 7.1. Let $\chi(s)$ be a smooth cut-off function of the real variable s satisfying

$$\chi(s) = \begin{cases} 1, & |s| \leq 1/2, \\ 0, & |s| \geq 1, \end{cases}$$

and such that $|\chi'(s)| \leq 3$ for all s , and $(1 - \chi^2)^{1/2} \in C^1(\mathbb{R})$. Next, let M and α be positive (to determined below) real numbers and define in \mathbb{R}^2 the functions χ_1 and χ_2 via $\chi_1(x) = \chi(MB^\alpha(1 - |x|))$ and $\chi_1(x)^2 + \chi_2(x)^2 = 1$. Then there exists a constant C_1 such that

$$\|\nabla \chi_j\|_\infty \leq C_1 M B^\alpha, \quad j \in \{1, 2\}.$$

Next, for $\ell > 0$, let $\Phi_\ell(x) = B^\sigma |1 - |x|| \chi(|x|/\ell)$. Then, pointwise in \mathbb{R} , it holds that $\Phi_\ell(x) \rightarrow B^\sigma |1 - |x||$ as $\ell \rightarrow +\infty$. Moreover, Φ_ℓ is differentiable almost everywhere and if $\ell \geq 2$ its gradient satisfies

$$\|\nabla \Phi_\ell\|_\infty \leq 4B^\sigma.$$

Moreover, Φ_ℓ is bounded for all $\ell > 0$, so the function $\Psi = \psi e^{\Phi_\ell}$ belongs to the form-domain of $\mathcal{H}(B)$.

With the IMS formula, we find that

$$\begin{aligned} \mathfrak{q}[\chi_1 \Psi] + \mathfrak{q}[\chi_2 \Psi] &\leq (2C_1 M^2 B^{2\alpha} + \lambda + 16B^{2\sigma}) \|\Psi\|^2 \\ &\leq (2C_1 M^2 B^{2\alpha} + \delta B + \omega B^{1/2} + 16B^{2\sigma}) \|\Psi\|^2. \end{aligned} \quad (7.3)$$

Using that the smallest Dirichlet eigenvalue is greater than the smallest value of the magnetic field (again, see [1]), we find that

$$\mathfrak{q}[\chi_1 \Psi] \geq \delta B \|\chi_1 \Psi\|^2$$

and

$$\mathfrak{q}[\chi_2 \Psi] \geq \left(\delta B + \frac{kB^{1-2\alpha}}{4M^2} \right) \|\chi_2 \Psi\|^2.$$

Inserting this into (7.3) we find that

$$\delta B \|\Psi\|^2 + \frac{kB^{1-2\alpha}}{4M^2} \|\chi_2 \Psi\|^2 \leq (2C_1 M^2 B^{2\alpha} + \delta B + \omega B^{1/2} + 16B^{2\sigma}) \|\Psi\|^2,$$

which can be written

$$\begin{aligned} \left(\frac{kB^{1-2\alpha}}{4M^2} - 2C_1 M^2 B^{2\alpha} - \omega B^{1/2} - 16B^{2\sigma} \right) \|\chi_2 \Psi\|^2 \\ \leq (2C_1 M^2 B^{2\alpha} + \omega B^{1/2} + 16B^{2\sigma}) \|\chi_1 \Psi\|^2. \end{aligned}$$

Choosing $\alpha = \sigma = \frac{1}{4}$, we find that all B s factor out, and hence

$$\left(\frac{k}{4M^2} - 2C_1 M^2 - \omega - 16 \right) \|\chi_2 \Psi\|^2 \leq (2C_1 M^2 + \omega + 16) \|\chi_1 \Psi\|^2.$$

With M so small that the left parenthesis above becomes positive, we find that there exists a constant C_2 such that

$$\|\chi_2 \Psi\|^2 \leq C_2 \|\chi_1 \Psi\|^2. \quad (7.4)$$

On the support of χ_1 it holds that $MB^{1/4}|1 - |x|| \leq 1$, and hence

$$\exp(\Phi_\ell) = \exp(B^{1/4}|1 - |x||\chi(|x|/\ell)) \leq \exp(\chi(|x|/\ell)/M) \leq \exp(1/M).$$

Inserting this in (7.4) above yields

$$\|\chi_2 \Psi\|^2 \leq C_2 \exp(2/M) \|\chi_1 \psi\|^2 \leq C_2 \exp(2/M) \|\psi\|^2.$$

Using monotone convergence we find that

$$\|\chi_2 \exp(B^{1/4}|1 - |x||) \psi\|^2 \leq C_2 \exp(2/M) \|\chi_1 \psi\|^2 \leq C_2 \exp(2/M) \|\psi\|^2.$$

On the other hand, since $MB^{1/4}|1 - |x|| \leq 1$ on the support of χ_1 it is clear that

$$\|\chi_1 \exp(B^{1/4}|1 - |x||) \psi\|^2 \leq \exp(2/M) \|\psi\|^2.$$

Combining these two last inequalities we find (7.1) with $C = (1 + C_2) \exp(1/M)$.

To prove (7.2) we essentially only have to reinsert the L^2 -estimate in the previous calculations. By monotone convergence and the IMS-formula, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \exp(2B^{1/4}|1 - |x||) |(-i\nabla + B\mathbf{F})\psi|^2 dx \\ = \lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^2} \exp(2\Phi_\ell) |(-i\nabla + B\mathbf{F})\psi|^2 dx \\ = \lim_{\ell \rightarrow \infty} \mathfrak{q}[\Psi] - \int |\nabla \Phi_\ell|^2 |\Psi|^2 dx. \end{aligned}$$

The last term is negative, and we can estimate the first term using again the IMS-formula and (7.3) as

$$\mathfrak{q}[\Psi] \leq \mathfrak{q}[\chi_1 \Psi] + \mathfrak{q}[\chi_2 \Psi] \leq (\delta B + C_2 B^{1/2}) \|\Psi\|^2$$

(with $C_2 = 2C_1 M^2 + \omega + 16$ and using $\alpha = \sigma = 1/4$). Now (7.2) follows from (7.1). \square

With the help of Proposition 7.1, we now get a first control of the involved angular momenta.

Lemma 7.3. *Let $\delta \geq 0$. Suppose $\psi = u_m e^{-im\theta}$ is an eigenfunction of $\mathcal{H}(B)$ with eigenvalue below $\delta B + \omega B^{1/2}$. Then*

$$m = \Phi B + \mathcal{O}(B^{3/4}).$$

The proof of Lemma 7.3 is similar to the one of Lemma 4.2—taking into account the weaker localization given by Proposition 7.1—and will be omitted.

7.3. A detailed expansion for $m - \Phi B = \mathcal{O}(B^{1/2})$. By Lemma 7.2 there is at most one eigenvalue of $\mathcal{H}_m(B)$ for sufficiently low energy. So it suffices to construct a trial state. The trial function (and all its derivatives) will be localized on the length scale $B^{-1/2}$ near $r = 1$ (see (7.10) for the explicit choice of trial state). Also the function has support away from $r = 0$. The calculation is slightly different in different regimes of angular momenta m . In this subsection, we consider angular momenta satisfying that

$$|m - \Phi B| \leq MB^{1/2}, \tag{7.5}$$

(for some fixed $M > 0$). The other case, where $MB^{1/2} \leq |m - \Phi B| \leq M'B^{3/4}$ is the object of the next subsection.

We will start by doing a formal expansion of the operator $\mathfrak{h} = \frac{1}{B} \mathcal{H}_m(B)$. We write

$$m = \Phi B + \mu_1 B^{1/2} + \mu_2.$$

With the localization of the trial state in mind, we introduce the new variable

$$\rho = (\delta B)^{1/2}(r - 1).$$

This leads to the expansion of our operator as in (4.7) but as operators on $L^2(\mathbb{R})$. Since in the present situation we do not have a boundary, we make the further translation $s := \rho - \mu_1/\sqrt{\delta}$ to find

$$\mathfrak{h} = \mathfrak{h}_0 + B^{-1/2}\mathfrak{h}_1 + B^{-1}\mathfrak{h}_2 + \dots$$

where

$$\begin{aligned} \mathfrak{h}_0 &= \delta \left(-\frac{d^2}{ds^2} + s^2 \right), \\ \mathfrak{h}_1 &= -\delta^{1/2} \frac{d}{ds} + s\delta^{-1/2} (\mu_1^2 - \delta(2\mu_2 + s^2)), \\ \mathfrak{h}_2 &= (s + \mu_1\delta^{-1/2}) \frac{d}{ds} + \mu_2^2 + \frac{(-\mu_1^2 + 3\delta s^2 + 2\delta^{1/2}\mu_1 s)}{\delta} \mu_2 \\ &\quad + \frac{(\mu_1 + \delta^{1/2}s)^2 (4ks(\mu_1 + \delta^{1/2}s) + 3\delta^{1/2}(\mu_1^2 - 6\delta^{1/2}s\mu_1 + 5\delta s^2))}{12\delta^{5/2}}. \end{aligned}$$

We do the same Ansatz as above and compare order by order:

Order B^0 : To leading order we find

$$\mathfrak{h}_0 v_0 = \lambda_0 v_0.$$

Thus, we choose

$$v_0 = \frac{1}{\pi^{1/4}} \exp(-s^2/2) \quad (7.6)$$

as the normalized ground state of the harmonic oscillator, and $\lambda_0 = \delta$.

Order $B^{-1/2}$: Here we get

$$(\mathfrak{h}_0 - \lambda_0)v_1 = (\lambda_1 - \mathfrak{h}_1)v_0.$$

By taking scalar product (with measure ds), we find

$$0 = \langle v_0, (\mathfrak{h}_0 - \lambda_0)v_1 \rangle = \lambda_1 - \langle v_0, \mathfrak{h}_1 v_0 \rangle.$$

Since v_0 is an even function it holds that $\langle v_0, \mathfrak{h}_1 v_0 \rangle = 0$ and thus $\lambda_1 = 0$. Moreover, since we can choose $v_1 \perp v_0$, we can let v_1 be the regularized resolvent $(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1}$ of $-\mathfrak{h}_1 v_0$,

$$v_1 = -(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1} [\mathfrak{h}_1 v_0]. \quad (7.7)$$

Order B^{-1} : We get

$$(\mathfrak{h}_0 - \lambda_0)v_2 = (\lambda_2 - \mathfrak{h}_2)v_0 + (\lambda_1 - \mathfrak{h}_1)v_1.$$

Taking scalar product with v_0 again and using the fact that $\lambda_1 = 0$, gives

$$\lambda_2 = \langle v_0, \mathfrak{h}_2 v_0 \rangle + \langle v_0, \mathfrak{h}_1 v_1 \rangle.$$

Now it holds that (remember: $v_0 = \frac{1}{\pi^{1/4}} \exp(-s^2/2)$)

$$\begin{aligned} \langle s v_0'(s), v_0(s) \rangle &= -\frac{1}{2}, & \langle s^2 v_0(s), v_0(s) \rangle &= \frac{1}{2}, \\ \langle v_0'(s), v_0(s) \rangle &= 0, & \langle s^4 v_0(s), v_0(s) \rangle &= \frac{3}{4}, \\ \langle s^j v_0(s), v_0(s) \rangle &= 0, \quad j \text{ odd}, & \langle s^6 v_0(s), v_0(s) \rangle &= \frac{15}{8}, \end{aligned}$$

and so

$$\langle v_0, \mathfrak{h}_2 v_0 \rangle = \frac{1}{4\delta^2} \mu_1^4 + \frac{2k - 3\delta - 4\delta\mu_2}{4\delta^2} \mu_1^2 + \mu_2^2 + \frac{3}{2} \mu_2 + \frac{7}{16} + \frac{k}{4\delta}. \quad (7.8)$$

The term $\langle v_0, \mathfrak{h}_1 v_1 \rangle$ is more difficult to calculate. But noting that

$$\begin{aligned} (\mathfrak{h}_0 - \lambda_0) \frac{1}{2\delta} s v_0 &= s v_0, \\ (\mathfrak{h}_0 - \lambda_0) \left(-\frac{1}{2\delta} s v_0 \right) &= v'_0, \quad \text{and} \\ (\mathfrak{h}_0 - \lambda_0) \frac{s(s^2 + 3)}{6\delta} v_0 &= s^3 v_0, \end{aligned}$$

we find that

$$\begin{aligned} v_1(s) &= -(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1} (\mathfrak{h}_1 v_0) \\ &= (\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1} \left(\delta^{1/2} v'_0(s) - \delta^{-1/2} \mu_1^2 s v_0(s) + 2\delta^{1/2} \mu_2 s v_0(s) + \delta^{1/2} s^3 v_0(s) \right) \\ &= -\frac{1}{2\delta^{1/2}} s v_0 - \frac{\mu_1^2}{2\delta^{3/2}} s v_0 + \frac{\mu_2}{\delta^{1/2}} s v_0 + \frac{s(s^2 + 3)}{6\delta^{1/2}} v_0. \end{aligned}$$

A direct calculation shows that

$$\begin{aligned} \mathfrak{h}_1 v_1 &= -\frac{s^2}{2\delta^2} \mu_1^4 + \frac{(4s^4 + 3(4\mu_2 - 1)s^2 + 3)}{6\delta} \mu_1^2 \\ &\quad + \frac{1}{6} (-6\mu_2 - s^6 + (1 - 8\mu_2)s^4 - 3(4\mu_2^2 - 2\mu_2 + 1)s^2) v_0, \end{aligned}$$

so, using the relations above, we find that

$$\langle v_0, \mathfrak{h}_1 v_1 \rangle = -\frac{1}{4\delta^2} \mu_1^4 + \frac{(4\mu_2 + 3)}{4\delta} \mu_1^2 - \frac{1}{16} (8\mu_2(2\mu_2 + 3) + 7). \quad (7.9)$$

Combining (7.8) and (7.9) we get

$$\lambda_2 = \langle v_0, \mathfrak{h}_2 v_0 \rangle + \langle v_0, \mathfrak{h}_1 v_1 \rangle = k \left(\frac{1}{2\delta^2} \mu_1^2 + \frac{1}{4\delta} \right).$$

We see that λ_2 is minimal when $\mu_1 = 0$.

Proof of Theorem 1.15. Using Proposition 7.5 below it suffices to consider angular momenta satisfying (7.5).

To finish the proof, based on the calculations above, it is sufficient to provide the trial state that gives the right energy. This is done as in the case of the exterior of the disc, see Section 4 for the details.

We write down the trial state (and λ) for the sake of completeness. From the calculations above it follows that (here $\mu_1 = 0$ and μ_2 is bounded)

$$\lambda = \lambda_0 + \lambda_1 B^{-1/2} + \lambda_2 B^{-1} = \delta + \frac{k}{4\delta} B^{-1}$$

Let v_0 be the gaussian given in (7.6), v_1 the function given in (7.7) and

$$v_2(s) = (\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1} [(\lambda_2 - \mathfrak{h}_2) v_0 + (\lambda_1 - \mathfrak{h}_1) v_1].$$

Next, let

$$v(s) = v_0 + v_1 B^{-1/2} + v_2 B^{-1}.$$

With $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi(0) = 1$ and $\varepsilon = 1/100$ we define our trial state $\tilde{v}(r)$ as

$$\tilde{v}(r) = B^{1/4} \chi(B^{1/2-\varepsilon}(r-1)) v((\delta B)^{1/2}(r-1)). \quad (7.10)$$

□

7.4. Excluding large values of $m - \Phi B$. In this subsection we will make a preliminary calculation to show that the ground state energy of $\mathcal{H}(B)$ restricted to angular momentum m is too large, unless $m - \Phi B = \mathcal{O}(B^{-1/2})$.

Lemma 7.4. *Let $C_0 > 0$, then there exists $C_1 > 0$ such that if $|m - \Phi B| \leq C_0 B^{3/4}$, then*

$$\text{dist}(\sigma(\mathcal{H}_m(B)), \delta B + f(\eta)\sqrt{B}) \leq C_1(|\eta|B^{-1/4} + B^{-1/2}).$$

Here $\eta := \frac{B\Phi - m}{\delta B^{3/4}}$, and

$$f(\eta) = \frac{1}{2}k\eta^2.$$

From Lemma 7.4 we can improve the localization in angular momentum.

Proposition 7.5. *Let $\omega > 0$. Then there exists $M, B_0 > 0$ such that if $B \geq B_0$ and $\mathcal{H}_m(B)$ has an eigenvalue below $\delta B + \omega$, then*

$$|m - \Phi B| \leq MB^{1/2}.$$

Proof of Proposition 7.5. This follows by combining Lemma 7.2 and 7.4. \square

Proof of Lemma 7.4. The proof is by trial state. We will construct a function (see specific choice in (7.12) below) $\varphi \in \text{Dom}(\mathcal{H}_m(B))$ such that $\|\varphi\| \approx 1$ and

$$\|(\mathcal{H}_m(B) - [\delta B + f(\eta)\sqrt{B}])\varphi\| \leq C_1(|\eta|B^{-1/4} + B^{-1/2}). \quad (7.11)$$

By the Spectral Theorem, this implies the Lemma, like in Section 4. The function that we construct will be localized near $r = 1$ on the length scale $B^{-1/2}$ (again this is exactly as in Section 4).

We recall that

$$\mathcal{H}_m(B) = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} (m - Bra(r))^2.$$

Here we will need to expand $\tilde{\beta}$ further than the second derivative, so we use the full expansion of $ra(r)$ from (2.4).

Introducing η as in the lemma and $\rho = (r - 1 + B^{-1/4}\eta)\sqrt{B}$, we find

$$\begin{aligned} \mathcal{H}_m(B) = & -B \frac{d^2}{d\rho^2} - \frac{\sqrt{B}}{1 - B^{-1/4}\eta + B^{-1/2}\rho} \frac{d}{d\rho} \\ & + \frac{1}{(1 - B^{-1/4}\eta + B^{-1/2}\rho)^2} \times \\ & \left[m - B(1 - B^{-1/4}\eta + B^{-1/2}\rho)a(1 - B^{-1/4}\eta + B^{-1/2}\rho) \right]^2. \end{aligned}$$

Since we will only act with $\mathcal{H}_m(B)$ on functions which in the ρ variable are Schwartz functions (see specific choice in (7.12) below), we can treat ρ as a quantity of order 1 (in terms of powers of B), and expand

$$\mathcal{H}_m(B) = B \left(\mathfrak{h}_{m,0} + B^{-1/4}\mathfrak{h}_{m,1} + B^{-1/2}\mathfrak{h}_{m,2} \right) + \mathcal{O}(|\eta|B^{1/2}) + \mathcal{O}(1),$$

where

$$\begin{aligned}\mathfrak{h}_{m,0} &= -\frac{d^2}{d\rho^2} + \delta^2 \left(\rho + \frac{\eta^2}{2} \right)^2, \\ \mathfrak{h}_{m,1} &= \frac{1}{3} \delta \eta^3 (3\delta - k) \left(\rho + \frac{\eta^2}{2} \right), \quad \text{and} \\ \mathfrak{h}_{m,2} &= -\frac{d}{d\rho} - \delta^2 \left(\rho + \frac{\eta^2}{2} \right)^3 + k \delta \eta^2 \left(\rho + \frac{\eta^2}{2} \right)^2 \\ &\quad + \frac{1}{12} \left(\delta \eta^4 (c - 7k + 15\delta) \right) \left(\rho + \frac{\eta^2}{2} \right) + \frac{1}{36} (k - 3\delta)^2 \eta^6.\end{aligned}$$

We choose

$$v_0 = \left(\frac{\delta}{\pi} \right)^{1/4} \exp \left(-\frac{\delta}{2} \left(\rho + \eta^2/2 \right)^2 \right),$$

which is the normalized ground state eigenfunction of $\mathfrak{h}_{m,0}$ with eigenvalue δ .

Next,

$$\mathfrak{h}_{m,1} v_0 = \frac{1}{3} \delta \eta^3 (3\delta - k) \left(\rho + \eta^2/2 \right) v_0.$$

Thus, we want to solve

$$(\mathfrak{h}_{m,0} - \delta) v_1 = -\mathfrak{h}_{m,1} v_0 = -\frac{1}{3} \delta \eta^3 (3\delta - k) \left(\rho + \eta^2/2 \right) v_0,$$

for v_1 . A calculation shows that (note that $\mathfrak{h}_{m,1} v_0$ is the first excited state of $\mathfrak{h}_{m,1}$ with eigenvalue 3δ , in particular orthogonal to v_0)

$$v_1 = -\frac{1}{2\delta} \mathfrak{h}_{m,1} v_0 = -\frac{1}{6} \eta^3 (3\delta - k) \left(\rho + \eta^2/2 \right) v_0$$

gives a solution.

Further calculations yields (the 0 is there since $\langle v_0, \mathfrak{h}_{m,1} v_0 \rangle = 0$)

$$\langle v_0, \mathfrak{h}_{m,2} v_0 \rangle + \langle v_0, (\mathfrak{h}_{m,1} - 0) v_1 \rangle = \frac{1}{2} k \eta^2.$$

We further choose

$$v_2 = -(\mathfrak{h}_{m,0} - \delta)_{\text{reg}}^{-1} [(\mathfrak{h}_{m,2} - f(\eta)) v_0 + \mathfrak{h}_{m,1} v_0].$$

With φ in (7.11) being chosen as

$$\varphi = (v_0 + B^{-1/4} v_1 + B^{-1/2} v_2) \times \chi(B^{1/2-\varepsilon}(r-1)), \quad (7.12)$$

(similarly to (4.6)), it is immediate to verify (7.11). \square

8. THE CASE OF THE WHOLE PLANE WITH $\delta = 0$

Here we will consider the case $\Omega = \mathbb{R}^2$ and a magnetic field β satisfying Assumption 1.5, with $\delta = 0$. We recall that in this section, $k > 0$.

By Proposition 7.1 and Lemma 7.3 we have localization of eigenfunctions corresponding to low-lying eigenvalues on the length scale $B^{-1/4}$ and to angular momenta $m = \frac{B}{12} + \mathcal{O}(B^{3/4})$.

By (2.2) and (2.3), for $|r-1| \leq 1$,

$$\begin{aligned}\left(\frac{m}{r} - Ba(r) \right)^2 &= \frac{1}{r^2} \left(m - \Phi B - \frac{Bk}{6} (r-1)^3 + B\mathcal{O}(|r-1|^4) \right)^2 \\ &\geq \frac{1}{2} \frac{(m - B\Phi)^2}{r^2} - CB^2 r^{-2} (r-1)^6.\end{aligned} \quad (8.1)$$

Invoking the localization estimates we get the following strengthening of Lemma 7.3.

Lemma 8.1. *Let $\delta = 0$. Suppose $\psi = u_m e^{-im\theta}$ is an eigenfunction of $\mathcal{H}(B)$ with eigenvalue below $\omega B^{1/2}$. Then*

$$m = \Phi B + \mathcal{O}(B^{1/4}).$$

Proof. The proof follows from inserting (8.1) in the formula for the quadratic form \mathfrak{q}_m and using the decay estimates in Proposition 7.1. \square

We also get a similar result to Lemma 4.3.

Lemma 8.2. *Let $\omega < \inf_{\alpha \in \mathbb{R}} \lambda_{2, \mathcal{H}_M(\alpha)}$, with $\mathcal{H}_M(\alpha)$ the operator from Appendix B. There exists $B_0 > 0$ such that if $m \in \mathbb{Z}$ and $B \geq B_0$, then \mathfrak{q}_m admits at most one eigenvalue below $(k/2)^{1/2} \omega B^{1/2}$.*

Proof. The proof is analogous to that of Lemma 4.3. By the localization estimates already obtained, we can see that \mathfrak{q}_m is given—up to a lower order error—by the quadratic form of the operator \mathfrak{h}_0 from (8.2) below which can be recognized as the ‘Montgomery’ operator reviewed in Appendix B. \square

So now we are again in a situation where we know that a sufficiently precise trial state must give the asymptotics of the ground state energy. We write

$$m = \Phi B + \mu_3 B^{1/4} + \mu_4$$

where we will keep μ_3 and μ_4 bounded. We perform the change of variables

$$\rho = B^{1/4}(r - 1).$$

Integrating by parts, we find (with $v(\rho) = B^{-1/8}u(1 + B^{-1/4}\rho)$ and assuming that u is supported away from 0) that

$$\frac{1}{B^{1/2}} \int_0^{+\infty} \left| \frac{du}{dr} \right|^2 r dr = \int_{-\sqrt[4]{B}}^{+\infty} \bar{v} \left(-\frac{d^2 v}{d\rho^2} - B^{-1/4}(1 + B^{-1/4}\rho)^{-1} \frac{dv}{d\rho} \right) (1 + B^{-1/4}\rho) d\rho.$$

We let $\mathfrak{h} = \frac{1}{B^{1/2}} \mathcal{H}_m(B)$ and make the Ansatz

$$\mathfrak{h} = \sum_{j=0}^{+\infty} \mathfrak{h}_j B^{-j/4}, \quad \lambda = \sum_{j=0}^{+\infty} \lambda_j B^{-j/4}, \quad \text{and} \quad v = \sum_{j=0}^{+\infty} v_j B^{-j/4},$$

and get (with notation from (2.4) and where $d = \tilde{\beta}^{(4)}(1)$)

$$\begin{aligned} \mathfrak{h}_0 &= -\frac{d^2}{d\rho^2} + \left(\frac{k\rho^3}{6} - \mu_3 \right)^2, \\ \mathfrak{h}_1 &= -\frac{d}{d\rho} - \left(\frac{k\rho^3}{6} - \mu_3 \right) \left(\frac{(k-c)\rho^4}{12} - 2\mu_3\rho + 2\mu_4 \right) \\ \mathfrak{h}_2 &= \rho \frac{d}{d\rho} + \mu_4^2 - 4\mu_3\mu_4\rho + 3\mu_3^2\rho^2 + \frac{1}{12}(5k-c)\mu_4\rho^4 \\ &\quad + \frac{1}{60}(6c-d-30k)\mu_3\rho^5 + \frac{1}{2880}(5c^2-18ck+8dk+45k^2)\rho^8. \end{aligned} \tag{8.2}$$

Next we compare the powers of B .

Order B^0 : We note that, after a scaling, \mathfrak{h}_0 becomes

$$\left(\frac{k}{2} \right)^{1/2} \left[-\frac{d^2}{d\rho^2} + \left(\frac{\rho^3}{3} - (2/k)^{1/4} \mu_3 \right)^2 \right] = \left(\frac{k}{2} \right)^{1/2} \mathcal{H}_M((2/k)^{1/4} \mu_3),$$

with the notation from Appendix B. By the results of the appendix, the ground state eigenvalue $\lambda_{1, \mathcal{H}_M(\alpha)}$ has a unique non-degenerate minimum Ξ at $\alpha = 0$. So we take $\mu_3 = 0$ and find that $\lambda_0 = (k/2)^{1/2} \lambda_{1, \mathcal{H}_M(0)} = (k/2)^{1/2} \Xi$. We furthermore take v_0 to be the ground state eigenfunction of \mathfrak{h}_0 (with $\mu_3 = 0$).

Order $B^{-1/4}$: Here the equation becomes

$$(\mathfrak{h}_1 - \lambda_1)v_0 + (\mathfrak{h}_0 - \lambda_0)v_1 = 0.$$

Taking scalar products with v_0 , we get

$$\lambda_1 = \langle v_0, \mathfrak{h}_1 v_0 \rangle = 0,$$

where we used that $\mu_3 = 0$ and that v_0 is an even function. We also determine the function v_1 as

$$v_1 = -(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1}(\mathfrak{h}_1 v_0)$$

Order $B^{-1/2}$: At this order, we consider the equation

$$(\mathfrak{h}_2 - \lambda_2)v_0 + \mathfrak{h}_1 v_1 + (\mathfrak{h}_0 - \lambda_0)v_2 = 0.$$

Taking scalar products with v_0 determines λ_2 ,

$$\lambda_2 = \langle v_0, \mathfrak{h}_2 v_0 \rangle + \langle v_0, \mathfrak{h}_1 v_1 \rangle.$$

As a function of μ_4 we see that λ_2 is a polynomial of degree 2. We determine the coefficient to μ_4^2 as

$$1 - 4 \left\langle \frac{k\rho^3}{6} v_0, (\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1} \frac{k\rho^3}{6} v_0 \right\rangle.$$

From perturbation theory, we recognize this expression as $\frac{1}{2} \frac{d^2}{d\alpha^2} \lambda_{1, \mathcal{H}_M(\alpha)}|_{\alpha=0}$, which is positive (by Theorem B.1 and Proposition B.3).

Thus

$$\lambda_2(\mu_4) = \frac{c_0}{2}(\mu_4 - C_1)^2 + C_2,$$

with $c_0 > 0$ and for suitable constants C_1, C_2 . We fix

$$v_2 = -(\mathfrak{h}_0 - \lambda_0)_{\text{reg}}^{-1}[(\mathfrak{h}_2 - \lambda_2)v_0 + \mathfrak{h}_1 v_1].$$

Proof of Theorem 1.16. To complete the proof of Theorem 1.16 we only need to give the trial state that gives the right energy. This is done in the same way as it was done for the complement of the disc in Lemma 4.5. We omit the details, but mention that the trial state is given by (here $\varepsilon = 1/100$ and $\chi \in C_0^\infty$ with $\chi(0) = 1$)

$$\tilde{v}(r) = B^{1/8} \chi(B^{\frac{1}{4}-\varepsilon}(r-1)) v(B^{1/4}(r-1)),$$

with $v = v_0 + B^{-1/4}v_1 + B^{-1/2}v_2$ from the calculations above. \square

Proof of Theorem 1.14. From Theorem 1.16 it follows exactly like in the proof of Theorem 1.8 that $B \mapsto \lambda_{1, \mathcal{H}(B)}$ is not monotone increasing on any half-interval of the form $[B_0, \infty)$. This finishes the proof of Theorem 1.14. \square

APPENDIX A. THE DE GENNES OPERATOR

In this section we have collected some known results on the one-dimensional self-adjoint operator

$$\mathcal{H}_{\text{dG}}(\xi) = -\frac{d^2}{d\rho^2} + (\rho - \xi)^2$$

in $L^2((0, +\infty))$ with Neumann condition at $\rho = 0$.

We denote by $\lambda_{1, \mathcal{H}_{\text{dG}}(\xi)}$ the lowest eigenvalue of $\mathcal{H}_{\text{dG}}(\xi)$ and let φ_ξ denote the (positive, normalized) ground state.

It is well-known (see for example [8]) that this eigenvalue has a unique minimum

$$\Theta_0 = \min_{\xi \in \mathbb{R}} \lambda_{1, \mathcal{H}_{\text{dG}}(\xi)},$$

attained at the unique positive

$$\xi_0 = (\Theta_0)^{1/2}.$$

Moreover, this minimum is non-degenerate; its second derivative at this point equals $2\xi_0\varphi_{\xi_0}(0)^2$. The following momentum formulas hold:

$$\begin{aligned}\langle \varphi_{\xi_0}, \varphi_{\xi_0} \rangle &= 1, & \langle \varphi_{\xi_0}, (\rho - \xi_0)\varphi_{\xi_0} \rangle &= 0, \\ \langle \varphi_{\xi_0}, (\rho - \xi_0)^2\varphi_{\xi_0} \rangle &= \frac{1}{2}\xi_0^2, & \langle \varphi_{\xi_0}, (\rho - \xi_0)^3\varphi_{\xi_0} \rangle &= \frac{1}{6}\varphi_{\xi_0}(0)^2.\end{aligned}\tag{A.1}$$

From these formulas we also find

$$\begin{aligned}\langle \varphi_{\xi_0}, \rho\varphi_{\xi_0} \rangle &= \xi_0, & \langle \varphi_{\xi_0}, \rho^2\varphi_{\xi_0} \rangle &= \frac{3}{2}\xi_0^2, \quad \text{and} \\ \langle \varphi_{\xi_0}, \rho^3\varphi_{\xi_0} \rangle &= \frac{1}{6}\varphi_{\xi_0}(0)^2 + \frac{5}{2}\xi_0^3.\end{aligned}$$

Moreover, it holds that

$$\langle \varphi_{\xi_0}, \varphi'_{\xi_0} \rangle = -\frac{1}{2}\varphi_{\xi_0}(0)^2.\tag{A.2}$$

If we denote by $(\mathcal{H}_{\text{dG}}(\xi_0) - \Theta_0)_{\text{reg}}^{-1}$ the regularized resolvent, then a straight forward calculation shows that

$$(\mathcal{H}_{\text{dG}}(\xi_0) - \Theta_0)_{\text{reg}}^{-1}[(\rho - \xi_0)\varphi_{\xi_0}] = -\frac{1}{2}\varphi'_{\xi_0} - \frac{1}{4}\varphi_{\xi_0}(0)^2\varphi_{\xi_0},$$

and hence (here we use one of the momentum relations above and integration by parts)

$$\begin{aligned}1 - 4\left\langle (\rho - \xi_0)\varphi_{\xi_0}, (\mathcal{H}_{\text{dG}}(\xi_0) - \Theta_0)_{\text{reg}}^{-1}[(\rho - \xi_0)\varphi_{\xi_0}] \right\rangle \\ = 1 - 4\left\langle (\rho - \xi_0)\varphi_{\xi_0}, -\frac{1}{2}\varphi'_{\xi_0} - \frac{1}{4}\varphi_{\xi_0}(0)^2\varphi_{\xi_0} \right\rangle \\ = 1 - 4\left\langle (\rho - \xi_0)\varphi_{\xi_0}, -\frac{1}{2}\varphi'_{\xi_0} \right\rangle \\ = \xi_0\varphi_{\xi_0}(0)^2.\end{aligned}\tag{A.3}$$

In particular this expression is positive.

APPENDIX B. A MONTGOMERY OPERATOR

For $\alpha \in \mathbb{R}$, we define the Montgomery type operator

$$\mathcal{H}_M(\alpha) = -\frac{d^2}{d\rho^2} + \left(\frac{\rho^3}{3} - \alpha\right)^2$$

as a self-adjoint operator on $L^2(\mathbb{R})$. Let us denote by

$$\lambda_{1, \mathcal{H}_M(\alpha)} < \lambda_{2, \mathcal{H}_M(\alpha)} \leq \dots$$

the eigenvalues of $\mathcal{H}_M(\alpha)$, with corresponding eigenfunctions u_1, u_2 and so on.

Theorem B.1 ([10]). *The function $\alpha \mapsto \lambda_{1, \mathcal{H}_M(\alpha)}$ has a unique minimum at $\alpha = 0$. Furthermore, the minimum is non-degenerate, i.e.*

$$c_0 := \frac{d^2}{d\alpha^2} \lambda_{1, \mathcal{H}_M(\alpha)} \Big|_{\alpha=0} > 0.\tag{B.1}$$

We introduce the following notation,

$$\Xi := \lambda_{1, \mathcal{H}_M(0)}.\tag{B.2}$$

Remark B.2. By the estimates in [10] we know that

$$0.618 \approx \frac{\sqrt{5} - 1}{2} < \Xi < \frac{2^{3/2}}{9} \left(\frac{4\pi^6 - 210\pi^4 + 4410\pi^2 - 26775}{7} \right)^{1/4} \approx 0.664.$$

A numerical value of Ξ , calculated by V. Bonnaillie-Noël, is 0.66.

Proposition B.3. *It holds that*

$$\frac{d^2}{d\alpha^2} \lambda_{1, \mathcal{H}_M(\alpha)} \big|_{\alpha=0} = 2 - 8 \left\langle \frac{\rho^3}{3} u_1, (\mathcal{H}_M(0) - \Xi)_{\text{reg}}^{-1} \frac{\rho^3}{3} u_1 \right\rangle.$$

Proof. Perturbation theory. \square

APPENDIX C. NUMERICAL CALCULATIONS

The eigenvalues of $\mathcal{H}_m(B)$ can be solved explicitly in terms of confluent hypergeometric functions, and plotted by the computer. Below we include a figure with the lowest eigenvalue of the limit operator $\mathcal{A}(B)$ and the lowest eigenvalue of the annulus of inner radius $R_i = 1$ and outer radius $R_o = 3/2$.

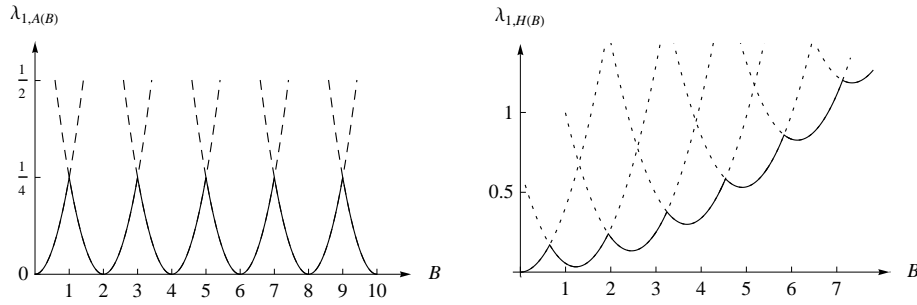


FIGURE 1. Left: The eigenvalues of $\mathcal{A}(B)$ (dotted) and the lowest eigenvalue $\lambda_{1,\mathcal{A}(B)}$ (solid) for $0 < B < 10$ and $R_i = 1$. Right: The lowest eigenvalue $\lambda_{1,\mathcal{H}(B)}$ plotted for $0 < B < 8$. The dotted lines are the lowest eigenvalue of $\mathcal{H}_m(B)$ for $0 \leq m \leq 6$, $R_i = 1$ and $R_o = 3/2$.

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(Søren Fournais) AARHUS UNIVERSITY, DEPARTMENT OF MATHEMATICS, NY MUNKEGADE 118, 8000 AARHUS C, DENMARK

E-mail address: `fournais@imf.au.dk`

(Mikael Persson Sundqvist) LUND UNIVERSITY, DEPARTMENT OF MATHEMATICAL SCIENCES, LUND, SWEDEN

E-mail address: `mickep@maths.lth.se`